University College of Cape Breton MATH115

MATRIX ALGEBRA

April 2003 Time : 3 hours

All questions carry an equal weight and can be attempted in any order. Clearly write your answers to the questions showing all working, explanation and checks used.

Q1. (a) Find the eigenvectors of $A := \begin{bmatrix} 13 & 64 & -16 \\ 8 & 29 & -8 \\ 40 & 160 & -43 \end{bmatrix}$ if it has eigenvalues 5 and -3. [7]

(b) Diagonalise A and check that AP is equal to PD for the appropriate P and D. [3]

Q2. Identify and prove which of the three vector space axioms are true for each of these sets and give counterexamples for those axioms which are false. [10]

$$\{(x, y, z) \mid |z - x| = y\}, \quad \{(x, y) \mid xy > 3\}, \quad \{(x, y, z) \mid z \geq 2y + 3x\}$$

Q3. (a) Prove that if $C = C^{-1}$ then $\det(C) = +1$ or -1. [1]

(b) Find the general form of 2×2 matrices which satisfy the equation $C = C^{-1}$. [6]

(c) Show that $L := \begin{bmatrix} -2/3 & 2/3 & 1/3 \\ 2/3 & 1/3 & 2/3 \\ 1/3 & 2/3 & -2/3 \end{bmatrix}$ is an orthogonal matrix. [2]

(d) Explain why L also satisfies $L = L^{-1}$. [1]

Q4. (a) Find the determinant of $B := \begin{bmatrix} 4 & 2 & -1 \\ -3 & -1 & x \\ 2 & 1 & 2 \end{bmatrix}$ using a Laplace expansion. [2]

(b) Find the inverse of B using the adjoint method. [6]

(c) Check the determinant of B^{-1} is equal to $(\det(B))^{-1}$. [2]

Q5. (a) From the set of vectors $\{v_1, v_2, v_3\} := \{\begin{pmatrix} -1 \\ 1 \\ -3 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 2 \\ 4 \end{pmatrix}\}$ get an orthogonal set of vectors using the Gram-Schmidt method. [7]

(b) Show that the set $\{v_1, v_2, v_3, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}\}$ is independent. [3]

Q6. (a) Find the eigenvectors of $E_1 := \begin{bmatrix} -23 & -15 \\ 50 & 32 \end{bmatrix}$ and $E_2 := \begin{bmatrix} 37 & 10 \\ -105 & -28 \end{bmatrix}$. [7]

(b) Deduce that these two matrices are similar and find P such that $E_2 = PE_1P^{-1}$. [3]

Q7. We are given the matrix $M := \begin{bmatrix} 5 & -5 & 11 & -2 & 14 & 4 \\ -6 & 6 & -16 & 1 & -20 & -5 \\ 2 & -2 & 10 & 2 & 12 & 2 \\ 3 & -3 & -27 & -18 & -30 & 0 \end{bmatrix}$

(a) Find all solutions to $Mx = \begin{bmatrix} -24 \\ 31 \\ -14 \\ 12 \end{bmatrix}$ [7]

(b) Deduce that the rank of M is 2 and give a basis for the column space of M. [3]

Q8. (a) Show that the plane $P := \{(x, y, z) \mid 7x - 5y - z = 0\}$ contains all points of the form $a \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix} + b \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ [2]

(b) Verify that every point on the line $\begin{bmatrix} -3t \\ -5t \\ 4t \end{bmatrix}$ is on P, find an orthogonal line which is also in the plane and express the two vectors in part (a) in terms of these two lines. [5]

(c) What line is the intersection of P with $Q := \{(x, y, z) \mid -4x + y + 3z = 0\}$? [3]