Cape Breton University

Math 1204

MATRIX ALGEBRA

Time : $\frac{3}{2}$ hours

February 2013

Please answer any THREE of these questions, please make sure to give all reasoning and working for all questions answered. Start a fresh sheet of paper for each question attempted.

Q1. (a) Evaluate $\det(A - \lambda I)$ using a cofactor expansions and ~~and use trial substitution~~ to get all three ~~3~~ eigenvalues. [6]

$$A := \begin{pmatrix} -22 & -16 & 4 \\ 39 & 27 & -6 \\ 0 & -4 & 4 \end{pmatrix}$$

(b) ~~Choose one of these eigenvalues (or ask me for one if you cannot find one) and~~ use it to create an eigenvector for A with no fractions in and find one of the other two eigenvectors [5]

$$\det(A-\lambda I) = \det\begin{pmatrix} -22-\lambda & -16 & 4 \\ 39 & 27-\lambda & -6 \\ 0 & -4 & 4-\lambda \end{pmatrix} = 4\times\det\begin{pmatrix} -22-\lambda & 4 \\ 39 & -6 \end{pmatrix} + (4-\lambda)\det\begin{pmatrix} -22-\lambda & -16 \\ 39 & 27-\lambda \end{pmatrix}$$

$$= 4\times(132+6\lambda-156) + (4-\lambda)(\lambda^2-5\lambda-594+624)$$

$$= 24\times(\lambda-4) + (4-\lambda)(\lambda^2-5\lambda+30)$$

$$= (4-\lambda)(\lambda^2-5\lambda+6) = (4-\lambda)(\lambda-3)(\lambda-2)$$

$$= -\lambda^3+9\lambda^2-26\lambda+24$$

so $\lambda_1 = 4$ $\lambda_2 = 3$ $\lambda_3 = 2$

$A-4I$:

$$\left(\begin{array}{ccc|c} -26 & -16 & 4 & 0 \\ 39 & 23 & -6 & 0 \\ 0 & -4 & 0 & 0 \end{array}\right)$$

$R_1 \leftarrow R_1 - 4R_3$

$R_2 \leftarrow R_2 + \frac{23}{4}R_3$

$$\left(\begin{array}{ccc|c} -26 & 0 & 4 & 0 \\ 39 & 0 & -6 & 0 \\ 0 & -4 & 0 & 0 \end{array}\right)$$

$R_2 \leftarrow R_2 + \frac{3}{2}R_1$

$-13x + 2z = 0$

$-4y = 0$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 13 \end{pmatrix}$$

$A-3I$:

$$\left(\begin{array}{ccc|c} -25 & -16 & 4 & 0 \\ 39 & 24 & -6 & 0 \\ 0 & -4 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} -25 & 0 & 0 & 0 \\ 39 & 0 & 0 & 0 \\ 0 & -4 & 1 & 0 \end{array}\right)$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$$

$A-2I$

$$\left(\begin{array}{ccc|c} -24 & -16 & 4 & 0 \\ 39 & 25 & -6 & 0 \\ 0 & -4 & 2 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} -24 & -8 & 0 & 0 \\ 39 & 13 & 0 & 0 \\ 0 & -4 & 2 & 0 \end{array}\right)$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1/3 \\ 3/3 \\ 6/3 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 6 \end{pmatrix}$$

$3x = -y$

$2z = 4y$

Q2. (a) Write these equations in matrix form and use row operations to get them to an equivalent of reduced row echelon form. What is the rank of the matrix? [6]

$$
\begin{aligned}
v + w + z &= 4\\
w + 2x + 2y + z &= 7\\
v + w - y + z &= 2\\
v - 2x + 2y &= 5\\
w + 2x + y + z &= 5
\end{aligned}
$$

(b) Find two homogeneous solutions which are not multiples of each other, ~~and check~~ and a combination with 3 zeros. Check yourself [4]

(c) Find ~~a particular solution with as many zeros as possible in, explaining why no more are possible.~~ Why is it impossible to have a partic sol with more than 2 zeros in? [1]

$$
\begin{array}{c}
\begin{array}{ccccc} v & w & x & y & z \end{array}\\
\left(\begin{array}{ccccc|c}
1 & 1 & 0 & 0 & 1 & 4\\
0 & 1 & 2 & 2 & 1 & 7\\
1 & 1 & 0 & -1 & 1 & 2\\
1 & 0 & -2 & 2 & 0 & 5\\
0 & 1 & 2 & 1 & 1 & 5
\end{array}\right)
\end{array}
$$

R_3 R_4

$$
\left(\begin{array}{ccccc|c}
1 & 1 & 0 & 0 & 1 & 4\\
0 & 1 & 2 & 2 & 1 & 7\\
0 & 0 & 0 & -1 & 0 & -2\\
0 & -1 & -2 & 2 & -1 & 1\\
0 & 1 & 2 & 1 & 1 & 5
\end{array}\right)
$$

R_2 R_4 R_5

$$
\left(\begin{array}{ccccc|c}
1 & 1 & 0 & 0 & 1 & 4\\
0 & 1 & 2 & 0 & 1 & 3\\
0 & 0 & 0 & -1 & 0 & -2\\
0 & -1 & -2 & 0 & -1 & -3\\
0 & 1 & 2 & 0 & 1 & 3
\end{array}\right)
$$

R_4 R_5

$$
\left(\begin{array}{ccccc|c}
1 & 1 & 0 & 0 & 1 & 4\\
0 & 1 & 2 & 0 & 1 & 3\\
0 & 0 & 0 & -1 & 0 & -2\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
$$

Rank = 3

$$
\begin{aligned}
v + w + z &= 4\\
w + 2x + z &= 3\\
y &= 2
\end{aligned}
$$

$$
\begin{aligned}
v &= 4 - w - z\\
2x &= 3 - w - z\\
y &= 2
\end{aligned}
$$

$$
\begin{pmatrix} v\\ w\\ x\\ y\\ z \end{pmatrix} = \begin{pmatrix} 4\\ 0\\ 3/2\\ 2\\ 0 \end{pmatrix} + \begin{pmatrix} -1\\ 1\\ -1/2\\ 0\\ 0 \end{pmatrix} w + \begin{pmatrix} -1\\ 0\\ -1/2\\ 0\\ 1 \end{pmatrix} z
$$

Note $w = -z$ gives $\begin{pmatrix} 0\\ 1\\ 0\\ 0\\ -1 \end{pmatrix}$

Q3. (a) Find the adjoint of B. [5]

$$B := \begin{pmatrix} 4 & x & -3 \\ y & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

deduce det(B)

(b) Multiply your answer by B to check and give B^{-1}. [3]

(c) Why is there no value for x or y such that B is guaranteed non-singular? [1]

Find two different x and y which give determinant 7. Why

$$\mathrm{adj}(B) = \begin{pmatrix} \det\begin{pmatrix}1&1\\1&0\end{pmatrix} & -\det\begin{pmatrix}y&1\\1&0\end{pmatrix} & \det\begin{pmatrix}y&1\\1&1\end{pmatrix} \\ -\det\begin{pmatrix}x&-3\\1&0\end{pmatrix} & \det\begin{pmatrix}4&-3\\1&0\end{pmatrix} & -\det\begin{pmatrix}4&x\\1&1\end{pmatrix} \\ \det\begin{pmatrix}x&-3\\1&1\end{pmatrix} & -\det\begin{pmatrix}4&-3\\y&1\end{pmatrix} & \det\begin{pmatrix}4&x\\y&1\end{pmatrix} \end{pmatrix}^T = \begin{pmatrix} -1 & -(-1) & y-1 \\ -(-3) & (-3) & -(4-x) \\ x+3 & -(4-3y) & 4-xy \end{pmatrix}^T$$

$$= \begin{pmatrix} -1 & -3 & x+3 \\ 1 & 3 & -4-3y \\ y-1 & x-4 & 4-xy \end{pmatrix}$$

$$B \times \mathrm{adj}(B) = \begin{pmatrix} 4 & x & -3 \\ y & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & -3 & x+3 \\ 1 & 3 & -4-3y \\ y-1 & x-4 & 4-xy \end{pmatrix} = \begin{pmatrix} -4+x-3y+3 & -12+3x-3x+12 & 4x+12-4x-3xy-12+3xy \\ -y+1+y-1 & -3y+3+x-4 & xy+3y-4-3y+4-xy \\ -1+1 & -3+3 & x+3-4-3y \end{pmatrix}$$

$$= \begin{pmatrix} x-3y-1 & 0 & 0 \\ 0 & x-3y-1 & 0 \\ 0 & 0 & x-3y-1 \end{pmatrix}$$

So $$B^{-1} = \frac{1}{x-3y-1}\begin{pmatrix} -1 & -3 & x+3 \\ 1 & 3 & -4-3y \\ y-1 & x-4 & 4-xy \end{pmatrix}$$

$\det(B) = x-3y-1 = 0$ when $x = 3y+1$

$x-3y-1 = 7$ when $x = 8+3y$

So $y=0$ $x=8$

or $y=-2$ $x=2$

Q4. (a) What are the eigenvalues and eigenvectors of the matrix $M := \begin{pmatrix} -3 & 4 \\ -1 & -8 \end{pmatrix}$? [3] 3

(b) Evaluate $M^2 + 28I$ ~~in terms of M~~ and relate it to M. [1]

(c) For real numbers a, b and c define $N := \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ and evaluate $N^2 + \det(N)I$. [4] 3

(d) Under what circumstances will N not have two eigenvectors? [1]
Why are a and c eigenvalues of N?

$\det\left(\begin{pmatrix} -3-\lambda & 4 \\ -1 & -8-\lambda \end{pmatrix}\right) = \lambda^2 + 11\lambda + 24 - -4 = \lambda^2 + 11\lambda + 28 = (\lambda+4)(\lambda+7)$

$M - -4I: \begin{pmatrix} 1 & 4 & : & 0 \\ -1 & -4 & ; & 0 \end{pmatrix} \quad \underline{v} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} \qquad M - -7I \begin{pmatrix} 4 & 4 & | & 0 \\ -1 & -1 & | & 0 \end{pmatrix} \quad \underline{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$M^2 = \begin{pmatrix} -3 & 4 \\ -1 & -8 \end{pmatrix}\begin{pmatrix} -3 & 4 \\ -1 & -8 \end{pmatrix} = \begin{pmatrix} 9-4 & -12-32 \\ 3+8 & -4+64 \end{pmatrix} = \begin{pmatrix} 5 & -44 \\ 11 & 60 \end{pmatrix}$ so $M^2 + 28I = \begin{pmatrix} 33 & -44 \\ 11 & 88 \end{pmatrix}$
$= -11M$

$N^2 = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a^2 & b(a+c) \\ 0 & c^2 \end{pmatrix} \qquad \det(N) = ac$

so $N^2 + ac\underline{I} = \begin{pmatrix} a^2+ac & b^2(a+c) \\ 0 & c^2+ac \end{pmatrix} = (a+c)\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$

Eigenvalues are a and c

so
$\underline{v}_a: \begin{pmatrix} 0 & b & ; & 0 \\ 0 & c-a & ! & 0 \end{pmatrix} \quad \underline{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \underline{v}_c \begin{pmatrix} a-c & b & ; & 0 \\ 0 & 0 & ; & 0 \end{pmatrix} \quad \underline{v} = \begin{pmatrix} b \\ c-a \end{pmatrix}$

But if a=c $\quad \underline{v}: \begin{pmatrix} 0 & b & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix}$ has only one solution unless b=0

END OF QUESTION PAPER