Q1)

$$\left(\begin{array}{cccc|c} 2 & -3 & 1 & 1 & 0 \\ 1 & -4 & 8 & 7 & -1 \\ -7 & 8 & 4 & 3 & -1 \\ -3 & 2 & 6 & 5 & -1 \end{array}\right)$$

$R_2 \leftarrow R_2 - 7R_1$
$R_3 \leftarrow R_3 - 3R_1$
$R_4 \leftarrow R_4 - 5R_1$

$$\left(\begin{array}{cccc|c} 2 & -3 & 1 & 1 & 0 \\ -13 & 17 & 1 & 0 & -1 \\ -13 & 17 & 1 & 0 & -1 \\ -13 & 17 & 1 & 0 & -1 \end{array}\right)$$

$R_1 \leftarrow R_1 - R_2$
$R_3 \leftarrow R_3 - R_2$
$R_4 \leftarrow R_4 - R_2$

$$\left(\begin{array}{cccc|c} 15 & -20 & 0 & 1 & 1 \\ -13 & 17 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$w = p$
$x = q$
R_2 $y = -1 + 13p - 17q$
R_1 $z = 1 - 15p + 20q$

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 13 \\ -15 \end{pmatrix} p + \begin{pmatrix} 0 \\ 1 \\ -17 \\ 20 \end{pmatrix} q$$

2) $\overset{w=}{p=3}$ $\overset{x=}{q=2}$ gives $y = -1+39-34 = 4$ and $z = 1-45+40 = -4$

$2\times3 - 3\times2 + 4 + -4 = 0$ ✓
$1\times3 - 4\times2 + 8\times4 + 7\times-4 = -5+4 = -1$ ✓
$-7\times3 + 8\times2 + 4\times4 + 3\times-4 = -5+4 = -1$ ✓
$-3\times3 + 2\times2 + 6\times4 + 5\times-4 = -5+4 = -1$ ✓

Need $p > q$, try $p = 4$ $q = 3$

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ -1+52-51=0 \\ 1-60+60 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$ ✓

Q2)

$$A=\begin{pmatrix} 4 & 2 & 3 \\ t & -4 & -3 \\ -1 & 2 & 3 \end{pmatrix} \quad \mathrm{Adj}(A)=\begin{pmatrix} \det\begin{pmatrix} -4 & -3 \\ 2 & 3 \end{pmatrix} & -\det\begin{pmatrix} t & -3 \\ -1 & 3 \end{pmatrix} & \det\begin{pmatrix} t & -4 \\ -1 & 2 \end{pmatrix} \\ -\det\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} & \det\begin{pmatrix} 4 & 3 \\ -1 & 3 \end{pmatrix} & -\det\begin{pmatrix} 4 & 2 \\ -1 & 2 \end{pmatrix} \\ \det\begin{pmatrix} 2 & 3 \\ -4 & -3 \end{pmatrix} & -\det\begin{pmatrix} 4 & 3 \\ t & -3 \end{pmatrix} & \det\begin{pmatrix} 4 & 2 \\ t & -4 \end{pmatrix} \end{pmatrix}^T$$

$$=\begin{pmatrix} -12+6 & -(3t-3) & -4+2t \\ -(6-6) & 12+3 & -(8--2) \\ -6+12 & -(-12-3t) & -16-2t \end{pmatrix}^T$$

$$=\begin{pmatrix} -6 & 3-3t & 2t-4 \\ 0 & 15 & -10 \\ 6 & 3t+12 & -2t-16 \end{pmatrix}^T$$

$$=\begin{pmatrix} -6 & 0 & 6 \\ 3-3t & 15 & 3t+12 \\ 2t-4 & -10 & -2t-16 \end{pmatrix}$$

$$A\,\mathrm{Adj}(A)=\begin{pmatrix} -24+6-6t+6t-12 & 30-30 & 24+6t+24-6t-48 \\ -6t-12+12t-3t+12 & -60+30 & 6t-12t-48+6t+48 \\ 6+6-6t+6t-12 & 30-30 & -6+6t+24-6t-48 \end{pmatrix}=\begin{pmatrix} -30 & 0 & 0 \\ 0 & -30 & 0 \\ 0 & 0 & -30 \end{pmatrix}$$

so $\det(A)=-30$

(b)

x	y	1		x	
	2	6			y
	1	3		0	0

Q3

$b_{n+1} = 1 \times b_n + 20 \times b_{n-1}$

$b_2 = -58 + 19 \times 20 = 322$

$\begin{pmatrix} b_{n+1} \\ b_n \end{pmatrix} = \begin{pmatrix} 1 & 20 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} b_1 \\ b_0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 20 \\ 1 & 0 \end{pmatrix}$

$\det(B - \lambda I) = \lambda^2 - \lambda - 20 = (\lambda - 5)(\lambda + 4)$ so $\lambda_1 = 5 \quad \lambda_2 = -4$

$\underline{v}_1 = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \quad \underline{v}_2 = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

so $\begin{pmatrix} b_{n+1} \\ b_n \end{pmatrix} = P D^n P^{-1} \begin{pmatrix} -58 \\ 19 \end{pmatrix}$

$= \begin{pmatrix} 5 & -4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5^n & 0 \\ 0 & (-4)^n \end{pmatrix} \frac{1}{9} \begin{pmatrix} 1 & 4 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} -58 \\ 19 \end{pmatrix}$

$= \begin{pmatrix} 5^{n+1} & (-4)^{n+1} \\ 5^n & (-4)^n \end{pmatrix} \begin{pmatrix} \frac{18}{9} \\ \frac{153}{9} \end{pmatrix} = \begin{pmatrix} 2 \times 5^{n+1} + 17 \times (-4)^{n+1} \\ 2 \times 5^n + 17 \times (-4)^n \end{pmatrix}$

so

$\underline{\underline{b_k = 2 \times 5^k + 17 \times (-4)^k}}$

$b_2 = 2 \times 25 + 17 \times 16 = 50 + 272 = 322$ ✓

b) For $b_k > 0$ we need $2 \times 5^k > 17 \times 4^k$ and k odd

$\left(\frac{5}{4}\right)^k > \frac{17}{2}$

$k > \frac{\log(\frac{17}{2})}{\log(\frac{5}{4})} = \frac{2.14}{0.223} = 9.59$

so try $k = 9$

$b_k = -550198$

Q4)

$$C_1 \leftarrow C_1 - C_3$$

$$\det(E-\lambda I) = \det\left(\begin{pmatrix} 21-\lambda & -8 & 16 \\ 16 & -3-\lambda & 16 \\ -16 & 8 & -11-\lambda \end{pmatrix}\right) = \det\left(\begin{pmatrix} 5-\lambda & -8 & 16 \\ 0 & -3-\lambda & 16 \\ \lambda-5 & 8 & -11-\lambda \end{pmatrix}\right)$$

$$R_3 \leftarrow R_3 + R_1$$

$$= \det\left(\begin{pmatrix} 5-\lambda & -8 & 16 \\ 0 & -3-\lambda & 16 \\ 0 & 0 & 5-\lambda \end{pmatrix}\right)$$

$$= (5-\lambda) \times \det\left(\begin{pmatrix} -3-\lambda & 16 \\ 0 & 5-\lambda \end{pmatrix}\right)$$

$$= (5-\lambda)(-3-\lambda)(5-\lambda)$$

so $\lambda_1 = \lambda_2 = 5 \quad \lambda_3 = -3$

$\underline{v}_3$

$$\left(\begin{array}{ccc|c} 24 & -8 & 16 & 0 \\ 16 & 0 & 16 & 0 \\ -16 & 8 & -8 & 0 \end{array}\right)$$

$R_1 \leftarrow R_1 \times \frac{1}{8}$

$R_2 \leftarrow R_2 \times \frac{1}{16}$

$R_3 \leftarrow R_3 \times \frac{-1}{8}$

$$\left(\begin{array}{ccc|c} 3 & -1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 \end{array}\right)$$

$R_1 \leftarrow R_1 - 2R_2$

$R_3 \leftarrow R_3 - R_2$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right)$$

$R_1 \leftarrow R_1 - R_3$

$R_3 \leftarrow -R_3$

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right)$$

$$\underline{v}_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$\underline{v}_1$ & $\underline{v}_2$:

$$\left(\begin{array}{ccc|c} 16 & -8 & 16 & 0 \\ 16 & -8 & 16 & 0 \\ -16 & 8 & -16 & 0 \end{array}\right)$$

$R_2 \leftarrow R_2 - R_1$

$R_3 \leftarrow R_3 + R_1$

$R_1 \leftarrow \frac{1}{8} R_1$

$$\left(\begin{array}{ccc|c} 2 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$y = 2x + 2z$

so

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}_{\underline{v}_1} x + \underbrace{\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}}_{\underline{v}_2} z$$