Test 3 2015

optional $c_2 \leftarrow c_2 + c_1$

(1)

Q1 (a) $\det(M-\lambda I) = \det\left(\begin{pmatrix} 11-\lambda & -28 & 8 \\ 10 & -22-\lambda & 4 \\ 2 & -2 & -7-\lambda \end{pmatrix}\right) = \det\left(\begin{pmatrix} 11-\lambda & -17-\lambda & 8 \\ 10 & -12-\lambda & 4 \\ 2 & 0 & -7-\lambda \end{pmatrix}\right)$

Laplace Row 3

Laplace (1) $= 2 \times \det\left(\begin{pmatrix} -17-\lambda & 8 \\ -12-\lambda & 4 \end{pmatrix}\right) - 0 \times \det(\sim) - (7+\lambda) \times \det\left(\begin{pmatrix} 11-\lambda & -17-\lambda \\ 10 & -12-\lambda \end{pmatrix}\right)$

2x2 det (1) $= 2 \times \left(-68-4\lambda+96+8\lambda\right) - (7+\lambda)\left(\lambda^2+12\lambda-11\lambda-132+170+10\lambda\right)$

expand (1) $= 2 \times (28+4\lambda) - (7+\lambda)(\lambda^2+11\lambda+38)$

$= 56 + 8\lambda - \lambda^3 - 11\lambda^2 - 38\lambda - 7\lambda^2 - 77\lambda - 266$

right answer (1) $= -\lambda^3 - 18\lambda^2 - 107\lambda - 210$

(1) $\lambda = 2$ $\quad -8-72-214-210 \neq 0$ as negative, need largish -ve λ to get 0

$\lambda = -5$ $\quad 125-450+535-210 = 660-660 = 0$

$$\begin{array}{r|l} & -\lambda^2 - 13\lambda - 42 \\ \lambda+5 & -\lambda^3 - 18\lambda^2 - 107\lambda - 210 \\ & \underline{-\lambda^3 - 5\lambda^2} \\ & \quad -13\lambda^2 - 107\lambda \\ & \quad \underline{-13\lambda^2 - 65\lambda} \\ & \qquad -42\lambda - 210 \\ & \qquad \underline{-42\lambda - 210} \\ & \qquad\qquad 0 \end{array}$$

(2)

(1) so $\det(M-\lambda I) = -(\lambda+5)(\lambda^2+13\lambda+42)$

$= -(\lambda+5)(\lambda+6)(\lambda+7)$

$\lambda = \frac{-13 \pm \sqrt{169-4\times 42}}{2} = \frac{-13 \pm 1}{2} = -6 \text{ or } -7$

or $(\lambda+6)(\lambda+7) = \lambda^2+6\lambda+7\lambda+42$

eigenvalues are $\lambda_1 = -5$ $\lambda_2 = -6$ $\lambda_3 = -7$.

$\underline{v}_1$

$$\left(\begin{array}{ccc|c} 16 & -28 & 8 & 0 \\ 10 & -17 & 4 & 0 \\ 2 & -2 & -2 & 0 \end{array}\right) \quad \left(\begin{array}{ccc|c} 24 & -36 & 0 & 0 \\ 14 & -21 & 0 & 0 \\ \cancel{4}\,2 & +2 & \cancel{+}2 & 0 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & -3/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \end{array}\right)$$

$R_1 \leftarrow R_1 + 4R_3$

$R_2 \leftarrow R_2 + 2R_3$

$\cancel{R_3 \leftarrow R_3 \times -\frac{1}{2}}$

$R_1 \leftarrow R_1 \times \frac{1}{24}$

$R_2 \leftarrow R_2 - 14 \times \text{new } R_1$

$R_3 \leftarrow R_3 \times -\frac{1}{2}$

$R_3 \leftarrow R_3 + R_1$

$$\left(\begin{array}{ccc|c} 1 & -3/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1/2 & 1 & 0 \end{array}\right)$$

$x = \frac{3}{2}t$

$y = t = \frac{2}{2}t$

$z = \frac{1}{2}t$

$\underline{v}_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

or

$\underline{v}_2$

$$\left(\begin{array}{ccc|c} 17 & -28 & 8 & 0 \\ 10 & -16 & 4 & 0 \\ 2 & -2 & -1 & 0 \end{array}\right) \quad \left(\begin{array}{ccc|c} 33 & -44 & 0 & 0 \\ 18 & -24 & 0 & 0 \\ 2 & -2 & -1 & 0 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & -4/3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & +2/3 & -1 & 0 \end{array}\right)$$

$R_1 \leftarrow R_1 + 8R_3$

$R_2 \leftarrow R_2 + 4R_3$

$R_1 \leftarrow R_1 \times \frac{1}{33}$

$R_2 \leftarrow R_2 \cancel{+} 18 \times R_1 \text{ new}$

$R_3 \leftarrow R_3 \cancel{+} 2 \times \text{new } R_1$

$x = \frac{4}{3}t$

$y = \frac{3}{3}t$

$z = \frac{2}{3}t$

$\underline{v}_2 = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$

or

$\underline{v}_3$

$$\left(\begin{array}{ccc|c} 18 & -28 & 8 & 0 \\ 10 & -15 & 4 & 0 \\ 2 & -2 & 0 & 0 \end{array}\right) \quad \left(\begin{array}{ccc|c} -2 & 2 & 0 & 0 \\ 10 & -15 & 4 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right) \quad \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -5 & 0 & 4 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right)$$

$R_1 \leftarrow R_1 - 2R_2$

$R_3 \leftarrow R_3 \times \frac{1}{2}$

$R_1 \leftarrow R_1 + 2R_3$

$R_2 \leftarrow R_2 - 10R_3$

$x = t = \frac{4}{4}t$

$\cancel{4}z = \frac{5}{4}t$

$y = x = \frac{4}{4}t$

$\underline{v}_3 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 5 \end{pmatrix}$

Q2)

(a) $\det(A-\lambda I) = (20-\lambda)(3-\lambda) - -5\times 14 = 60 - 23\lambda + \lambda^2 + 70$ ①

$= \lambda^2 - 23\lambda + 130$

$= (\lambda - 13)(\lambda - 10) = \lambda^2 + 13\lambda + 10\lambda + 130$ ✓

$\lambda_1 = +13 \quad \lambda_2 = +10$ ①

① $\underline{v}_1$ $\left(\begin{array}{cc:c} 20-13 & -5 & 0 \\ 14 & 3-13 & 0 \end{array}\right)$

$\left(\begin{array}{cc:c} 7 & -5 & 0 \\ 14 & -10 & 0 \end{array}\right)$ $R_2 \leftarrow R_2 - 2R_1$ $\quad \underline{v}_1 = \underline{\begin{pmatrix} 5 \\ 7 \end{pmatrix}}$

① $\underline{v}_2$ $\left(\begin{array}{cc:c} 20-10 & -5 & 0 \\ 14 & 3-10 & 0 \end{array}\right)$ $\left(\begin{array}{cc:c} 10 & -5 & 0 \\ 14 & -7 & 0 \end{array}\right)$ $R_2 \leftarrow R_2 - \frac{7}{5}R_1$ $\quad 10x - 5y = 0$

$y = 2x = 2t$

$\underline{v}_2 = \begin{pmatrix} x \\ y \end{pmatrix} = \underline{\begin{pmatrix} 1 \\ 2 \end{pmatrix}} t$

(b) $A^k = PD^kP^{-1} = \begin{pmatrix} 5 & 1 \\ 7 & 2 \end{pmatrix} \times \begin{pmatrix} 13^k & 0 \\ 0 & 10^k \end{pmatrix} \times \frac{1}{10-7} \begin{pmatrix} 2 & -1 \\ -7 & 5 \end{pmatrix}$ ①

$= \frac{1}{3} \times \begin{pmatrix} 5\times 13^k & 10^k \\ 7\times 13^k & 2\times 10^k \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -7 & 5 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 10\times 13^k - 7\times 10^k & -5\times 13^k + 5\times 10^k \\ 14\times 13^k - 14\times 10^k & -7\times 13^k + 10\times 10^k \end{pmatrix}$

$= \begin{pmatrix} \frac{10}{3}\times 13^k - \frac{7}{3}\times 10^k & -\frac{5}{3}\times 13^k + \frac{5}{3}\times 10^k \\ \frac{14}{3}\times 13^k - \frac{14}{3}\times 10^k & -\frac{7}{3}\times 13^k + \frac{10}{3}\times 10^k \end{pmatrix}$ ①

① $13 > 10$ so $13^k > 10^k$ for $k \geq 1$

largest coefficient of 13^k is $\frac{14}{3}$

so we want $\frac{14}{3} \times 13^k \geq 10^7$

$13^k \geq \frac{3\times 10^7}{14}$

$k \geq \frac{\log\left(\frac{3}{14}\times 10^8\right)}{\log 13} = 4.78$

① $k = 5 \quad \frac{14}{3}\times 13^5 - \frac{14}{3}\times 10^5 = 1266034$