Math205 Handout 2: Methods of Proof

We shall show how to go about proving the statement "if an odd number is multiplied by -1 and that new number is then subtracted from 26 the result is odd".

We first identify the statements $p(x)$ and $q(x)$ in the above statement and deduce that:

$$\begin{array}{lllll} p(x) & :\equiv & \text{"}x \text{ is odd "} & :\equiv & \text{"}x = 2j+1 \text{ for some } j \in \mathbb{Z}'' \\ q(x) & :\equiv & \text{"}26-x \text{ is odd"} & :\equiv & \text{"}26-x = 2k+1 \text{ for some } k \in \mathbb{Z}\text{"} \end{array}$$

We are usually either told to use one of the methods below, or we can choose one:

- **Direct:** We suppose that $p(x)$ is true and using what that tells us about x we then apply that to the subject of $q(x)$ in order to try to show that it is true when $p(x)$ is.

  So if $p(x)$ is true then $x = 2j+1$, and $q(x)$ is about $26-x$, and putting these two things together

  $26-x = 26-(2j+1) = 26-2j-1 = 25-2j = 25+2\times(-j) = 1+24+2\times(-j) = 2\times(12-j)+1$

  Thus we have our statement in the form of $q(x)$ where $k = 12-j$, and it just remains to establish that this k is an integer. Since j is, multiplying by -1 means $-j$ is still an integer, and then subtracting this from 12, another integer, means that k is an integer as required.

- **Contrapositive:** We can alternatively suppose that $q(x)$ is false and using what that tells us about x we then apply that to the subject of $p(x)$ in order to try to show that it is false when $p(x)$ is. We are thus proving $(\sim q(x)) \to (\sim p(x))$ which we know is logically equivalent to $p(x) \to q(x)$.

  So if $q(x)$ is false then $26-x = 2m, m \in \mathbb{Z}$, and $\sim p(x)$ says that "$x = 2n, n \in \mathbb{Z}$. Simplifying $(\sim q(x))$ to tell us about x

  $$\begin{aligned} 26-x &= 2m \\ x &= 26-2m \\ &= 2\times(13-m) \end{aligned}$$

  This equals $2n$ if we take $n = 13-m$ and so, again, since m is an integer, so is $-m$ and adding 13 to this keeps it an integer, so n is an integer as required.

- **Contradiction:** We now suppose that $p(x)$ is true and also that $q(x)$ is false. We intend to get an impossible situation arising whence we can use the logical equivalence of $(p(x) \wedge (\sim q(x))) \leftrightarrow (\sim T_0)$ and $(p(x) \to q(x)) \leftrightarrow T_0$ to show that $p(x)$ implies $q(x)$ as required.

  As before, if $p(x)$ is true then $x = 2j+1$, and $(\sim q(x))$ says that $26-x = 2m$. Combining these two statements to remove x we get:

  $$\begin{aligned} 26-(2j+1) &= 2m \\ 25 &= 2j+2m \\ &= 2(j+m) \\ \frac{25}{2} &= j+m \end{aligned}$$

  This statement is our desired contradiction since both j and m are integers and so their sum is an integer, but $\frac{25}{2}$ is certainly not an integer as it is 12.5 in decimal terms and no integer has to be written with a decimal point.