Math105 Handout 3: Induction and Sequences

- **Induction:** Given a statement $p(n)$ about an integer n we wish to show it is true for all integer values of n at least a and we proceed as follows:
  - *Initial Case:* Show that $p(a)$ is true (optionally also test $p(a+1)$ and $p(a+2)$ to see how the induction will proceed).
  - *Inductive Case:* Assume $p(k)$ is true for some value of $k \geq a$. State one side of $p(k+1)$ in terms of the corresponding side of $p(k)$ and use the assumptions to deduce that the other side of $p(k+1)$ is related in the same way as $p(n)$ was.

For example: $p(n) :=$ "$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$"

  - *Initial Case:* The first possible value of n is 1, so we consider $p(1) :=$ "$1^2 = 1 = \frac{1\times(1+1)\times(2\times 1+1)}{6} = 1$" as required. Similarly, $p(2) :=$ "$1^2+2^2 = 5 = \frac{2\times(2+1)\times(2\times 2+1)}{6} = 5$" and $p(3) :=$ "$1^2+2^2+3^2 = 14 = \frac{3\times(3+1)\times(2\times 3+1)}{6} = 14$".
  - *Inductive Case:* Assume $p(n) :=$ "$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$". Now

$$\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n+1)^2 = \text{lhs}(p(n)) + (n+1)^2.$$

But using the assumption (the inductive hypothesis), we get that

$$\begin{aligned}\sum_{i=1}^{n+1} i^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= (n+1)\frac{n(2n+1)+6(n+1)}{6} \\ &= (n+1)\frac{(2n^2+7n+6)}{6} \\ &= (n+1)\frac{(2n+3)(n+2)}{6} \\ &= \frac{(n+1)(n+2)(2n+3)}{6} \\ &= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}\end{aligned}$$

But this is exactly the statement $p(n+1)$ that we wished to establish!

- **Sigma Notation:**

$$1 + \ldots + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad 1^2 + \ldots + n^2 = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$b + \ldots + b = \sum_{i=1}^{n} b = nb \qquad 1 + x + \ldots + x^n = \sum_{i=0}^{n} x^i = \frac{x^{n+1} - 1}{x - 1}$$

- **Sequences:**
  - The sequence $a_{n+1} = ka_n + la_{n-1}$ is solved by following the following steps:
    - Solve the quadratic equation $x^2 - kx - l = 0$ to get roots s_1 and s_2.
    - If $s_1 \neq s_2$ then try the solution $a_n = u(s_1)^n + v(s_2)^n$ otherwise try $a_n = (u + nv)(s_1)^n$
    - Use your solution together with the given values for a_1 and a_2 to find the values of u and v.
    - Check that the recurrence relation is satisfied.
  - The sequence $a_{n+1} = ka_n + l$ has solution $a_n = u(k^n) + v$ which can be found similarly.
  - For a general recurrence relation:
    - Substitute the recurrence expression for a_n into the given expression for a_{n+1} and then repeat with the expression for a_{n-1} and a_{n-2} if necessary.
    - Guess a general formula for a_n in terms of a_1 and express any "..." as an algebraic expression using a formula from below.
    - Verify this closed solution against the recurrence relation and the expression for a_1. If an error exists then try a new guess.

- **Pigeonhole:**

  Given a set of n objects and m groups to place them in, if $m < n$ then at least one group has two objects in it. In general, we can say that one group must have at least

$$\left\lfloor \frac{n-1}{m} \right\rfloor + 1$$

  objects in it where the brackets round down the nearest integer.