Math205 Handout 4: Counting and Graphs

- **Counting:** The most general cases of counting involve choosing r objects from a pool of n objects under certain conditions. We have formulae for each of these combinations of conditions;
  - *Repetition*: whether or not we can draw "the same object" again once it has been drawn or not, as if we are replacing the object back in the pool of n objects each time we pick. Note that if we have multiple objects which are indistinguishable and we aren't replacing them, we are still in this case, assuming we aren't going to run out of these similar objects.
  - *Order*: whether or not we are picking all r objects at once or in a sequence, such as drawing a hand of cards or picking a soccer team by position.

	Repetition Allowed	Repitition Forbidden
Order Important	n^r a sequence of r rolls of a die n is number of faces of the die	$\dfrac{n!}{(n-r)!}$ r lotto balls in order n is number of balls
Order Unimportant	$\dbinom{r+n-1}{r} = \dfrac{(r+n-1)!}{r!(n-1)!}$ r arrows, how many in each segment? n is number of segments	$\dbinom{n}{r} = {}_nC_r = \dfrac{n!}{r!(n-r)!}$ hand of r cards n is number of cards

- **Relations:** We say that a set of ordered pairs (s, t), where $s \in S$ and $t \in T$ is a relation which is:

  Everywhere Defined: if there is a pair of the form $(s, ?)$ for all $s \in S$
  Onto: if there is a pair of the form $(?, t)$ for all $t \in T$
  Uniquely Defined: if there are no two pairs both of the form $(s, ?)$ for any $s \in S$
  One to One: if there are no two pairs both of the form $(?, t)$ for any $t \in T$

  A *function* is a relation which is both uniquely and everywhere defined. The inverse of a relation is the set of reversed ordered pairs (t, s). The inverse of a function is a function only if all 4 conditions hold.

  A relation R between a set S and itself is *reflexive* if $(s, s) \in R$ for all $s \in S$, it is *symmetric* if $(s, t) \in R$ implies $(t, s) \in R$ and *transitive* if $(s, t) \in R$ and $(t, u) \in R$ implies $(s, u) \in R$.

- **Graphs:** We have a set V of *vertices*, which are joined by a set E of *edges* between pairs of vertices forming a graph G. The *valency* of a vertex is the number of edges at it. The *valency sequence* is the valencies of each vertex arranged in non-increasing order and it tells us quite a lot about the graph. However, two graphs can be different and have the same valency sequence. We say that two graphs are *isomorphic* if there exists a 1-1 and onto function between the two sets of vertices which preserves all the edges. Normally we can see it more clearly by redrawing one graph to look like the other.

  We can move around the graph from vertex to vertex using the edges, as if they were towns and roads. Define a graph as *disconnected* if we cannot move along the edges to get from some vertex to another vertex, and *1-connected* if it is not disconnected, but there is a vertex which can be removed with its edges so that the remaining graph is disconnected.

  A *cycle* in a graph is a sequence of edges which returns to the start point without repeating any other vertices. If we can move through every vertex exactly once and return to the start the graph is *Hamiltonian*. It is difficult to tell when a graph is not Hamiltonian, but to show it is we can just demonstrate a cycle. If we can move along every edge exactly once (repeating vertices if necessary) and return to the start, the graph is *Eulerian* and this is true if and only if each vertex is of even valency. We can colour the vertices of a graph such that no edges joins two vertices of the same colour, and the smallest number of colours necessary for this is called its *colouring number*.