Math 205 Handout 3: Induction and Pigeonholes

- **Induction:** Given a statement p(n) about an integer n we wish to show it is true for all integer values of n at least a and we proceed as follows:
 - Initial Case: Show that p(a) is true (optionally also test p(a+1) and p(a+2) to see how the induction will proceed).
 - Inductive Case: Assume p(k) is true for some value of $k \ge a$. State one side of p(k+1) in terms of the corresponding side of p(k) and use the assumptions to deduce that the other side of p(k+1) is related in the same way as p(n) was.

For example:
$$p(n) := \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

- Initial Case: The first possible value of n is 1, so we consider $p(1) := \text{``}1^2 = 1 = \frac{1 \times (1+1) \times (2 \times 1+1)}{6} = 1$ " as required. Similarly, $p(2) := \text{``}1^2 + 2^2 = 5 = \frac{2 \times (2+1) \times (2 \times 2+1)}{6} = 5$ " and $p(3) := \text{``}1^2 + 2^2 + 3^2 = 14 = \frac{3 \times (3+1) \times (2 \times 3+1)}{6} = 14$ ".
- Inductive Case: Assume $p(k) := \sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$. Now the left hand side of p(k+1) is

$$\sum_{i=1}^{k+1} i^2 = \left(\sum_{i=1}^k i^2\right) + (k+1)^2 = LHS(p(k)) + (k+1)^2.$$

But using the assumption (the inductive hypothesis), we get that

$$\sum_{i=1}^{k+1} i^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= (k+1)\frac{k(2k+1) + 6(k+1)}{6}$$

$$= (k+1)\frac{(2k^2 + 7k + 6)}{6}$$

$$= (k+1)\frac{(2k+3)(k+2)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{(k+1)((k+1) + 1)(2(k+1) + 1)}{6}$$

But this is exactly the statement p(k+1) that we wished to establish!

• Basic formulae in Sigma Notation:

$$1 + \ldots + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad b + \ldots + b = \sum_{i=1}^{n} b = nb \qquad 1 + x + \ldots + x^{n} = \sum_{i=0}^{n} x^{i} = \frac{x^{n+1} - 1}{x - 1}$$

• Pigeonhole Principle:

Given a set of n objects and m groups to place them in, if m < n then at least one group has two objects in it. In general, we can say that one group must have at least

$$\left|\frac{n-1}{m}\right|+1$$

objects in it where the brackets round down the nearest integer.