Cape Breton University

Math205

DISCRETE MATH

December 2010

Time : 3 hours

Answer FIVE of the SEVEN questions, giving all working, reasoning and checks for your answers. The parts of the questions are weighted as shown and the questions can be answered in any order. Please start a fresh side of paper for each question.

Q1. The wheel graph W_n is defined (for $n \geq 4$) as the graph with n vertices formed by taking a cycle graph with $n-1$ vertices and adding one vertex joined to all others.

(a) Draw W_4, W_5 and W_6 and determine the connectedness and colouring numbers for W_n for any $n \geq 4$. [7]

(b) Considering the wheel graph now as a relation, explain why $W_n \circ W_n$ has every possible arrow. Give an example of a graph G which has a large connectedness but for which $G \circ G$ does not have every possible edge, explaining why. [5]

Q2. (a) Given this universal set of words $\mathcal{U} :=$ {sheep, deer, wolf, crow, camel, pigeon, aardvark, llama, cockatoo, elk, loon, gull, trout, ox, eel, stoat}, form a Venn diagram of the sets S and T defined in the following way: [3]

$$S := \{\text{words which contain a letter appearing twice}\}$$
$$T := \{\text{words with fewer than 5 letters in them}\}$$

(b) Count the number of ways to choose a set of 4 different words from $S \cup T$. [2]

(c) How many different ways are there to make a list of three words from $S \cap T^c$? Writing these lists in alphabetical order, count the number of lists until the place where a word first appears after a shorter word than itself. [3]

(d) How many different words must you choose from S before you are sure to have two with the same number of letters? How many different words must be chosen from S before you must have three words with the same number of letters? [3]

(e) What does the pigeonhole principle have to say about the minimum number of elements there must be in one of any two subsets of $\mathcal{U}$? [1]

Q3. (a) Simplify this logic statement to as few symbols as possible: [6]

$$((\sim p) \wedge (p \to q)) \vee ((q \to r) \wedge (\sim (r \wedge q)))$$

(b) Define $p(x) :\equiv \{x \in \mathbb{R} \mid x^2 + 3x > 4\}$ and $q(x) :\equiv \{x \in \mathbb{Z} \mid |2x+2| \leq 5\}$ and plot the areas in which they are true on the real line. Is it true or false that $\forall x \in \mathbb{Z}\ ;\ p(x) \to (\sim q(x))$? $\exists y \in \mathbb{R}\ ;\ p(y) \wedge q(y)$? $\exists w < 0\ ;\ p(w) \vee q(w)$? [6]

Q4. (a) You are given that M is a relation from J to K. Prove directly that if M is everywhere defined and one-to-one then $|J| \leq |K|$. [5]

(b) Give an example of an everywhere defined relation E from $J := \{1, 3, 4, 6, 7\}$ to $K := \{\alpha, \beta, \gamma\}$ and determine what pairs of elements are in $E^{-1} \circ E$ and $E \circ E^{-1}$. Are either of these an equivalence relation? Under what circumstances will an equivalence relation be formed from composing a relation and its inverse in general? [7]

Q5. (a) Prove by induction that: [8]

$$\sum_{j=3}^{n} (j^2 - 3j - 2) = \frac{(n-2)(n+3)(n-4)}{3}$$

(b) Without redoing the proof, explain why $\sum_{j=1}^{n} (j^2 - 3j - 2) = \sum_{j=6}^{n} (j^2 - 3j - 2)$ and express this sum as a factorised expression. [4]

Q6. (a) Explain why a graph with n vertices cannot have a vertex of valency $n-1$ and a vertex of valency 0 at the same time. Use that statement and the pigeonhole principle to prove that any graph must have two vertices of the same valency. [6]

(b) Create a Hamiltonian graph with 7 vertices and as many different valencies as possible, justifying why you have a best example. How little repetition of valencies can there be in an Eulerian graph? [6]

Q7. (a) Prove by induction that the number of subsets of a set of cardinality d is 2^d. [6]

(b) List all nine subsets of a set which are intersections of some combination of the subsets A, B, A^c and B^c. Explain why there are 3^m sets which can be expressed as intersections of at most m subsets and their complements. Count with $m = 3$ how many sets involve each of the different feasible numbers of subsets. [6]

END OF QUESTION PAPER