# Math205 Handout 3: Counting and Graphs

- **Pigeonhole Principle:**

  Given a set of $p$ objects and $h$ groups to place them in, if $h < p$ then at least one group has two objects in it. In general, we can say that one group must have at least

  $$\left\lceil \frac{p}{h} \right\rceil$$

  objects in it where those straight brackets round up the nearest integer.

  Note that it can sometimes be useful to round down, in which case the formula is $\left\lfloor \frac{p-1}{h} \right\rfloor + 1$

- **Counting:** The most general cases of counting involve choosing $r$ objects from a pool of $n$ objects under certain conditions. We have formulae for each of these combinations of conditions;

  - *Repetition*: whether or not we can draw "the same object" again once it has been drawn or not, as if we are replacing the object back in the pool of $n$ objects each time we pick. Note: if we have multiple objects which are indistinguishable and we aren't replacing them, we are still in this case, *assuming* we aren't going to run out of these similar objects.
  - *Order*: whether or not we are picking all $r$ objects at once or in a sequence, such as drawing a hand of cards or picking a soccer team by position.

| | Repetition Allowed | Repitition Forbidden |
|---|---|---|
| Order Important | $n^r$<br>a sequence of $r$ rolls of a die<br>$n$ is number of faces of the die | ${}_nP_r = \dfrac{n!}{(n-r)!}$<br>$r$ lotto balls in order<br>$n$ is number of balls |
| Order Unimportant | $\binom{r+n-1}{r} = \dfrac{(r+n-1)!}{r!(n-1)!}$<br>$r$ arrows, how many in each segment?<br>$n$ is number of segments | $\binom{n}{r} = {}_nC_r = \dfrac{n!}{r!(n-r)!}$<br>hand of $r$ cards<br>$n$ is number of cards |

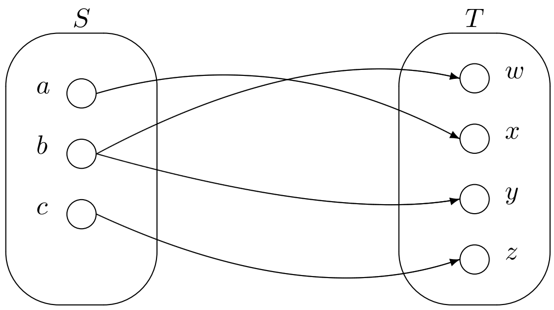
- **Relations:** We say that a set of ordered pairs $(s,t)$, where $s \in S$ and $t \in T$ is a relation which is:

  *Everywhere Defined:* if there is a pair of the form $(s,y)$ for all $s \in S$

  *Onto:* if there is a pair of the form $(x,t)$ for all $t \in T$

  *Uniquely Defined:* if there are no two pairs both of the form $(s,y)$ for any $s \in S$

  *One to One:* if there are no two pairs both of the form $(x,t)$ for any $t \in T$




$$R := \{(a,x),(b,w),(c,z),(b,y)\}$$

This $R$ is a relation from $S := \{a,b,c\}$ to $T := \{w,x,y,z\}$ which is e.d. but not u.d. since there are two arrows from $b$. $R$ is onto and 1-1 since there is exactly one arrow into each element in $T$.

A relation $R$ between a set $S$ and itself is *reflexive* if $(s, s) \in R$ for all $s \in S$, it is *symmetric* if $(s, t) \in R$ implies $(t, s) \in R$ and *transitive* if $(s, t) \in R$ and $(t, u) \in R$ implies $(s, u) \in R$. A relation is *anti-symmetric* if $(s, t) \in R$ implies $(t, s) \notin R$. A relation is *anti-reflexive* if $\forall s \in S; (s, s) \notin R$.

- A *function* is a relation which is both uniquely and everywhere defined.
- The *inverse of a relation* is the set of reversed ordered pairs $(t, s)$. Note that the inverse of a function is a function only if it is onto and 1-1.
- A relation which is reflexive, symmetric and transitive is an *equivalence relation.* An example of this is "$s - t$ is even" on the integers.
- A relation which is reflexive, anti-symmetric and transitive is a *partial order.* An example of a total order is "$s \leq t$" on real numbers.
- A relation which is anti-reflexive and symmetric is a *graph*; see the final section for details.

- **Graphs:** We have a set $V$ of *vertices*, which are joined by a set $E$ of *edges* between pairs of vertices forming a graph $G$. The *valency* of a vertex is the number of edges at it. The *valency sequence* is the valencies of each vertex arranged in non-increasing order and it tells us quite a lot about the graph. However, two graphs can be different and have the same valency sequence. We say that two graphs are *isomorphic* if there exists a 1-1 and onto function between the two sets of vertices which preserves all the edges. Normally we can see it more clearly by redrawing one graph to look like the other.

  We can move around the graph from vertex to vertex using the edges, as if they were towns and roads. Define a graph as *disconnected* if we cannot move along the edges to get from some vertex to another vertex, and *1-connected* if it is not disconnected, but there is a vertex which can be removed with its edges so that the remaining graph is disconnected.

  For these two graphs they both have valency sequence $(4, 3, 3, 2, 2, 1, 1)$. However, $G$ and $H$ are not isomorphic. We can see that the vertex of valency 4 is a cut-vertex in both graphs but its removal splits the remainder of the graph into different sized subgraphs. Also note that it is not adjacent to a vertex of valency 1 in $H$.

  A *cycle* in a graph is a sequence of edges which returns to the start point without repeating any other vertices. If we can move through every vertex exactly once and return to the start the graph is *Hamiltonian.* It is difficult to tell when a graph is not Hamiltonian, but to show it is we can just demonstrate a cycle. If we can move along every edge exactly once (repeating vertices if necessary) and return to the start, the graph is *Eulerian* and this is true if and only if each vertex is of even valency.

  We can colour the vertices of a graph such that no edges joins two vertices of the same colour, and the smallest number of colours necessary for this is called its *colouring number.* Any graph without odd cycles can be proven to have colouring number at most two and, famously proven in the 1970s, any graph which can be drawn without any edges crossing must have colouring number at most four.