Cape Breton University MATH 1204

Matrix Algebra

April 2012 Time : 3 hours

Please answer any FIVE of these questions, please make sure to give all reasoning and working for all questions answered. Start each question on a new piece of paper.

Q1. We are given these a line and a plane as follows:

$$L_1 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} := \begin{pmatrix} -4 \\ 3 \\ 4 \end{pmatrix} \times t + \begin{pmatrix} 7 \\ 2 \\ -1 \end{pmatrix} \quad , \quad P : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \circ \begin{pmatrix} 9 \\ -2 \\ 3 \end{pmatrix} = -4$$

(a) At which point do L_1 and P intersect? Check this point lies in the plane. [3]

(b) What is the shortest distance between L_1 and $L_2 := \begin{pmatrix} -3 \\ 5 \\ 5 \end{pmatrix} \times s + \begin{pmatrix} -1 \\ 62 \\ -11 \end{pmatrix}$? [9]

Q2. (a) Find two non-parallel eigenvectors of matrix M that have -1 as an eigenvalue: [5]

$$M := \begin{pmatrix} 2 & -3 & -3 & 3 \\ 6 & -8 & -5 & 7 \\ 0 & -1 & 0 & 1 \\ 6 & -8 & -4 & 7 \end{pmatrix}$$

(b) Use Gram-Schmidt to get an orthogonal pair of eigenvectors for eigenvalue -1. [2]

(c) Use carefully chosen determinant row and column operations to create zeros in $\det(M - \lambda I)$ and hence or otherwise find the other two eigenvalues of M. [5]

Q3. (a) For which x is $J := \begin{pmatrix} -2 & -5 & x \\ 3 & -1 & -6 \\ 4 & -7 & -3 \end{pmatrix}$ a singular matrix? [3]

(b) Assuming x is not this value, use the adjoint method on the matrix with x in to find the general inverse of J and check your answer by multiplying it with J. [5]

(c) Do a cofactor expansion to get $\det(J^{-1})$ and simplify to get the expected value. [4]

Q4. (a) Solve this matrix equation in terms of X, assuming inverses exist and stating which properties of matrix algebra you used: [4]

$$2(B^T + 4A^{-1}X)^{-1} = BA$$

(b) Check if your solution is all integers for $A := \begin{pmatrix} 3 & 4 \\ 2 & -3 \end{pmatrix}$ and $B := \begin{pmatrix} 3 & -7 \\ -5 & 11 \end{pmatrix}$ [4]

(c) If EF is a square matrix, what can you say about the sizes of E and F? How can EF have an inverse if E and F are not square matrices? Explain why, in this case, only one of EF or FE can have an inverse, giving an example. [4]

Q5. (a) Solve this simultaneous pair of recurrences using diagonalisation; [10]

$$c_{n+1} := \frac{1}{143}\left(67c_n + 33d_n\right) \ , \quad d_{n+1} := \frac{1}{143}\left(44c_n - 54d_n\right) \ , \quad c_0 = 15 \ , \quad d_0 = 18$$

(b) What will the ratio $\frac{c_k}{d_k}$ tend to as k tends to ∞? How rapidly would you expect $\frac{c_k}{d_k}$ to move towards this value? Why? [2]

Q6. Let H be the hyperplane which has the following basis:

$$v_1 := \begin{pmatrix} 2 \\ 2 \\ 5 \\ 2 \end{pmatrix} \ , \quad v_2 := \begin{pmatrix} 6 \\ 3 \\ 6 \\ 7 \end{pmatrix} \ , \quad v_3 := \begin{pmatrix} 4 \\ 2 \\ 2 \\ 2 \end{pmatrix} \ ,$$

(a) Use row operations to show that $\underline{u} := \begin{pmatrix} -10 \\ -5 \\ 4 \\ 2 \end{pmatrix}$ is not in H. [4]

(b) Re-use the same row operations to show that $\underline{w} := \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$ is in H. [4]

(c) Express H in 4-dimensional dot product form by finding the vector normal to each vector in the basis and verify that $\underline{u}$ and $\underline{w}$ have the appropriate dot products with the normal, explaining why they should have those values. [4]

Q7. (a) What is the equation of the best fit straight line for this data? [4]

x	-2	3	5
y	-6	4	1

(b) What is the general equation of all *cubic* polynomials which fit exactly through these three points? [6]

(c) Why do none of these cubics have all their coefficients as integers? Explain why, nonetheless, y will be an integer for every integer x for every cubic with an integer coefficient of x^3. [2]

END OF QUESTION PAPER