Q1) (a) $M\begin{pmatrix}1\\5\\-2\end{pmatrix} = \begin{pmatrix}-5+10-2\\72-65+8\\-216+250-40\end{pmatrix} = \begin{pmatrix}3\\15\\-6\end{pmatrix} = 3\times\begin{pmatrix}1\\5\\-2\end{pmatrix}$ so $\lambda_1 = 3$

(b) $\det(M-\lambda I) = \det\left(\begin{pmatrix}-5-\lambda & 2 & 1\\72 & -13-\lambda & -4\\-216 & 50 & 20-\lambda\end{pmatrix}\right)$

$= (-5-\lambda)(\lambda^2-20\lambda+13\lambda-260+200) - 2\times(1440-72\lambda-864)$

$+ (3600-216\lambda-2808)$

$= (-5-\lambda)(\lambda^2-7\lambda-60) - 1152 + 144\lambda - 216\lambda + 792$

$= -\lambda^3+7\lambda^2-5\lambda^2+60\lambda+35\lambda+300-360-72\lambda$

$= -\lambda^3+2\lambda^2+23\lambda-60$

check $\lambda=3$ $-27+18+69-60=0$

$$
\begin{array}{r l}
 & \lambda^2+\lambda-20 \\
-\lambda+3 & \overline{)\,-\lambda^3+2\lambda^2+23\lambda-60} \\
 & \underline{-\lambda^3+3\lambda^2} \\
 & -\lambda^2+23\lambda \\
 & \underline{-\lambda^2+3\lambda} \\
 & 20\lambda-60 \\
 & \underline{20\lambda-60} \\
 & 0
\end{array}
$$

and $\lambda^2+\lambda-20 = (\lambda+5)(\lambda-4)$
$= \lambda^2+5\lambda-4\lambda-20$ ✓

so $\lambda_2=-5$ $\lambda_3=+4$

(c) $(M+5I \,\vdots\, 0) = \left(\begin{array}{ccc|c}0 & 2 & 1 & 0\\72 & -8 & -4 & 0\\-216 & 50 & 25 & 0\end{array}\right)$

$R_2 \leftarrow R_2 + 4R_1$
$R_3 \leftarrow R_3 - 25R_1$

$\left(\begin{array}{ccc|c}0 & 2 & 1 & 0\\18 & 0 & 0 & 0\\-216 & 0 & 0 & 0\end{array}\right)$

$R2 \leftarrow R2 \times \frac{1}{18}$
$R3 \leftarrow R3 + 216\times \text{new } R2$

$\left(\begin{array}{ccc|c}0 & 2 & 1 & 0\\1 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{array}\right)$

$x=0$
$z=-2y$
$y=t$

so $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0y\\1y\\-2y\end{pmatrix} = \begin{pmatrix}0\\1\\-2\end{pmatrix}y$

Q2

(a) $A^TXA^{-1} = (3I+B)A^T$ R distrib

$= 3IA^T + BA^T$ L Inverse

$= 3A^T + BA^T$

$\times$ right by A

$A^TXA^{-1}A = (3A^T + BA^T)A$

R distrib / L inverse

$A^TXI = 3A^TA + BA^TA$

R identity / Mult left $(A^T)^{-1}$

$(A^T)^{-1}A^TXI = (A^T)^{-1}(3A^TA + BA^TA)$

Distrib

$= (A^T)^{-1}(3A^TA) + (A^T)^{-1}BA^TA$

Scalar commut

$IX = 3(A^T)^{-1}A^TA + (A^T)^{-1}BA^TA$ left inverse

$= 3IA + (A^T)^{-1}BA^TA$ left identity

$$\underline{X = 3A + (A^T)^{-1}BA^TA}$$

(b) $A = \begin{pmatrix}3 & 4\\-2 & -3\end{pmatrix}$ $A^{-1} = \frac{1}{-9-8}\begin{pmatrix}-3 & -4\\2 & 3\end{pmatrix} = \begin{pmatrix}3 & 4\\-2 & -3\end{pmatrix} = A$

$B = \begin{pmatrix}2 & 1\\5 & 0\end{pmatrix}$ $A^T = \begin{pmatrix}3 & -2\\4 & -3\end{pmatrix} = (A^T)^{-1}$ $A^TA = \begin{pmatrix}13 & 18\\+18 & 25\end{pmatrix}$

$X = 3A + (A^T)^{-1}BA^TA = \begin{pmatrix}9 & 12\\-6 & -9\end{pmatrix} + \begin{pmatrix}-4 & 3\\-7 & 4\end{pmatrix}\begin{pmatrix}13 & 18\\+18 & 25\end{pmatrix}$

$= \begin{pmatrix}9 & 12\\-6 & -9\end{pmatrix} + \begin{pmatrix}2 & 3\\-19 & -26\end{pmatrix} = \underline{\begin{pmatrix}11 & 15\\-25 & -35\end{pmatrix}}$

$\cancel{A^TXA = (832\ \ldots)\begin{pmatrix}3 & 4\\-2 & 3\end{pmatrix}}$

(c) $C = \begin{pmatrix}a & b\\c & d\end{pmatrix}$ $C^T = \begin{pmatrix}a & c\\b & d\end{pmatrix}$ $C^{-1} = \frac{1}{ad-bc}\begin{pmatrix}d & -c\\-b & a\end{pmatrix}$

If $C=C^T$ and $C=C^{-1}$ then $b=c$ and $ad-b^2=-1$ and $a=-d$

so $a^2+b^2=1$ so $a=0$ $b=\pm 1$

or $a=\pm 1$ $b=0$

Q3

(a) $\det(E) = 5\times(xy-1)+1\times(-6y-8)-5\times(-6-8x)$

$$= 5xy - 5 - 6y - 8 + 30 + 40x$$

$$= 5xy - 6y + 40x + 17$$

(b) $\det(E) = 0$ when $(5x-6)y = -40x-17$

$$y = \frac{-40x-17}{5x-6} \quad \text{unless } 5x-6=0 \quad x=\frac{6}{5}$$

$$\det(E) = 6y - 6y + 48 + 17 = 65$$

When $x=1$ $y = \frac{-57}{-1} = 57$

(c)
$$\operatorname{adj}\left(\begin{pmatrix} 5 & -1 & -5 \\ -6 & x & 1 \\ 8 & 1 & -8 \end{pmatrix}\right) = \begin{pmatrix} -8x-1 & -(48-8) & -6-8x \\ -(8-5) & 0 & -(5--8) \\ -1-5x & -(5-30) & 5x-6 \end{pmatrix}^T = \begin{pmatrix} -8x-1 & -40 & -6-8x \\ -13 & 0 & -13 \\ 5x-1 & 25 & 5x-6 \end{pmatrix}^T$$

$$= \begin{pmatrix} -8x-1 & -13 & 5x-1 \\ -40 & 0 & 25 \\ -8x-6 & -13 & 5x-6 \end{pmatrix}$$

Check
$$\begin{pmatrix} 5 & -1 & -5 \\ -6 & x & 1 \\ 8 & 1 & -8 \end{pmatrix}\begin{pmatrix} -8x-1 & -13 & 5x-1 \\ -40 & 0 & 25 \\ -8x-6 & -13 & 5x-6 \end{pmatrix}$$

$$= \begin{pmatrix} -40x-5+40+40x+30 & -65+65 & 25x-5-25-25x+30 \\ 48x+6-40x-8x-6 & 78-13 & -30x+6+25x+5x-6 \\ -64x-8-40+64x+48 & -104+104 & 40x-8+25-40x+48 \end{pmatrix}$$

$$= \begin{pmatrix} 65 & 0 & 0 \\ 0 & 65 & 0 \\ 0 & 0 & 65 \end{pmatrix}$$

so $E^{-1} = \frac{1}{65}\begin{pmatrix} 8x+1 & 13 & 1-5x \\ 40 & 0 & -25 \\ 8x+6 & 13 & 6-5x \end{pmatrix}$

Q4) (a) $\det(F-\lambda I) = \det\left(\begin{pmatrix} 4-\lambda & -11 \\ 5 & -12-\lambda \end{pmatrix}\right)$

$= (4-\lambda)(-12-\lambda) - -55$

$= \lambda^2 + 12\lambda - 4\lambda - 48 + 55$

$= \lambda^2 + 8\lambda + 7$

$= (\lambda+7)(\lambda+1)$ so $\lambda_1 = -7$ $\lambda_2 = -1$

$\underline{v}_1 : \left(\begin{array}{cc|c} 11 & -11 & 0 \\ 5 & -5 & 0 \end{array}\right)$ $\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\underline{v}_2 : \left(\begin{array}{cc|c} 5 & -11 & 0 \\ 5 & -11 & 0 \end{array}\right)$ $\underline{v}_2 = \begin{pmatrix} 11 \\ 5 \end{pmatrix}$

(b) $F^{-1} = \frac{1}{-48+55}\begin{pmatrix} -12 & 11 \\ -5 & 4 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} -12 & 11 \\ -5 & 4 \end{pmatrix}$

$F^{-1}\underline{v}_1 = \frac{1}{7}\begin{pmatrix} -12+11 \\ -5+4 \end{pmatrix} = \frac{-1}{7}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ so eigenvalue is $\frac{-1}{7}$

$F^{-1}\underline{v}_2 = \frac{1}{7}\begin{pmatrix} -132+55 \\ -55+20 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 77 \\ 35 \end{pmatrix} = \begin{pmatrix} 11 \\ 5 \end{pmatrix} = 1 \times \begin{pmatrix} 11 \\ 5 \end{pmatrix}$ so eigenvalue is 1

$F^T = \begin{pmatrix} 4 & 5 \\ -11 & -12 \end{pmatrix}$ $F^T\underline{v}_1 = \begin{pmatrix} 4+5 \\ -11-12 \end{pmatrix} = \begin{pmatrix} 9 \\ -23 \end{pmatrix}$ $\frac{9}{-23} \neq \frac{1}{1}$

$F^T\underline{v}_2 = \begin{pmatrix} 44+25 \\ -121-60 \end{pmatrix} = \begin{pmatrix} 69 \\ -181 \end{pmatrix}$ $\frac{69}{-181} \neq \frac{11}{5}$

$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $M^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ $\det(M-\lambda I) = \lambda^2 - (a+d)\lambda + ad - bc$

$\det(M^T - \lambda I) = \lambda^2 - (d+a)\lambda + ad - cb$

Roots are same so so are eigenvalues

$\lambda_1 = -7$

$F^T + 7I : \left(\begin{array}{cc|c} 11 & 5 & 0 \\ -11 & -5 & 0 \end{array}\right)$ $\underline{w}_1 = \begin{pmatrix} -5 \\ 11 \end{pmatrix}$ $F^T + I : \left(\begin{array}{cc|c} 5 & 5 & 0 \\ -11 & -11 & 0 \end{array}\right)$ $\underline{w}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$