2016: 1204 Test 3 Question 1

a) $A-\lambda I = \begin{pmatrix} -66-\lambda & -40 \\ 104 & 63-\lambda \end{pmatrix}$ $\quad \det(A-\lambda I) = \lambda^2 + 66\lambda - 63\lambda - 4158 + 4160$

$$= \lambda^2 + 3\lambda + 2$$
$$= (\lambda+2)(\lambda+1)$$

$$\begin{array}{r} 66 \\ 63 \\ \hline 378 \\ 3780 \\ \hline 4158 \end{array}$$

so eigenvalues are -2 and -1

eigenvectors:

$\lambda_1 = -2$ $\quad \left(\begin{array}{cc|c} -64 & -40 & 0 \\ 104 & 65 & 0 \end{array}\right) \xrightarrow{} \left(\begin{array}{cc|c} 8 & 5 & 0 \\ 8 & 5 & 0 \end{array}\right)$ $\quad \underline{v}_1 = \begin{pmatrix} -5 \\ 8 \end{pmatrix}$

$R1 \times -\frac{1}{8}$ $\quad R2 \times \frac{1}{13}$ $\qquad R2 \leftarrow R2 - R1$

$\lambda_2 = -1$ $\quad \left(\begin{array}{cc|c} -65 & -40 & 0 \\ 104 & 64 & 0 \end{array}\right) \xrightarrow{} \left(\begin{array}{cc|c} 13 & 8 & 0 \\ 13 & 8 & 0 \end{array}\right)$ $\quad \underline{v}_2 = \begin{pmatrix} -8 \\ 13 \end{pmatrix}$

$R1 \times -\frac{1}{5}$ $\quad R2 \times \frac{1}{8}$ $\qquad R2 \leftarrow R2 - R1$

so $A^n = \begin{pmatrix} -5 & -8 \\ 8 & 13 \end{pmatrix} \begin{pmatrix} (-2)^n & 0 \\ 0 & (-1)^n \end{pmatrix} \frac{1}{\cancel{}} \begin{pmatrix} 13 & 8 \\ -8 & -5 \end{pmatrix}$

$$= \begin{pmatrix} -5\times(-2)^n & -8\times(-1)^n \\ 8\times(-2)^n & 13\times(-1)^n \end{pmatrix} \begin{pmatrix} -13 & -8 \\ +8 & +5 \end{pmatrix}$$

$$= \begin{pmatrix} -64\times(-1)^n + 65\times(-2)^n & -40\times(-1)^n + 40\times(-2)^n \\ 104\times(-1)^n - 104\times(-2)^n & 65\times(-1)^n - 64\times(-2)^n \end{pmatrix}$$

b) $A^n \times \begin{pmatrix} 8 \\ -13 \end{pmatrix} = \begin{pmatrix} -5\times(-2)^n & -8\times(-1)^n \\ 8\times(-2)^n & 13\times(-1)^n \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -8\times(-1)^n \\ 13\times(-1)^n \end{pmatrix} = \begin{pmatrix} 8 \\ -13 \end{pmatrix}$ or $\begin{pmatrix} -8 \\ 13 \end{pmatrix}$

any multiple of $\underline{v}_2$

Question 2

(a) $M = \begin{pmatrix} 2 & 9 & -18 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $\quad \det(M - \lambda I) = \lambda^3 - 2\lambda^2 - 9\lambda + 18 = 0$

$\lambda = 1 \quad 1 - 2 - 9 + 18 = 8$

(b)

$\lambda = 2 \quad 8 - 8 - 18 + 18 = 0 \qquad \lambda - 2$

$\lambda = 3 \quad 27 - 18 - 27 + 18 = 0 \qquad \lambda - 3$

$$\begin{array}{r|l} & \lambda + 3 \\ \lambda^2 - 5\lambda + 6 & \lambda^3 - 2\lambda^2 - 9\lambda + 18 \\ & \lambda^3 - 5\lambda^2 + 6\lambda \\ \hline & 3\lambda^2 - 15\lambda + 18 \\ & 3\lambda^2 - 15\lambda + 18 \\ \hline & 0 \end{array}$$

$\lambda = -3 \quad -27 - 18 + 27 + 18 = 0$

$\underline{v}_1 = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} \quad \underline{v}_2 = \begin{pmatrix} 9 \\ 3 \\ 1 \end{pmatrix} \quad \underline{v}_3 = \begin{pmatrix} 9 \\ -3 \\ 1 \end{pmatrix}$

$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{pmatrix} \quad P = \begin{pmatrix} 4 & 9 & 9 \\ 2 & 3 & -3 \\ 1 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 2^5 & 3^5 & (-3)^5 \end{pmatrix}$

(c) $\begin{pmatrix} b_{j+1} \\ b_j \\ b_{j-1} \end{pmatrix} = M \begin{pmatrix} b_j \\ b_{j-1} \\ b_{j-2} \end{pmatrix} = M^{j-1} \begin{pmatrix} b_2 \\ b_1 \\ b_0 \end{pmatrix} = \begin{pmatrix} 4 & 9 & 9 \\ 2 & 3 & -3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2^{j-1} & 0 & 0 \\ 0 & 3^{j-1} & 0 \\ 0 & 0 & (-3)^{j-1} \end{pmatrix} \begin{pmatrix} 4 & 9 & 9 \\ 2 & 3 & -3 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 89 \\ 19 \\ 26 \end{pmatrix}$

$\underline{w}$: $\left(\begin{array}{ccc|c} 4 & 9 & 9 & 89 \\ 2 & 3 & -3 & 19 \\ 1 & 1 & 1 & 26 \end{array}\right) \left(\begin{array}{ccc|c} 0 & 5 & 5 & -15 \\ 0 & 1 & -5 & -33 \\ 1 & 1 & 1 & 26 \end{array}\right) \left(\begin{array}{ccc|c} 0 & 0 & 30 & 150 \\ 0 & 1 & -5 & -33 \\ 1 & 0 & 6 & 59 \end{array}\right) \left(\begin{array}{ccc|c} 0 & 0 & 1 & 5 \\ 0 & 1 & -5 & -33 \\ 1 & 0 & 6 & 59 \end{array}\right)$

$\left(\begin{array}{ccc|c} 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & -8 \\ 1 & 0 & 0 & 29 \end{array}\right)$ so $\underline{w} = \begin{pmatrix} 29 \\ -8 \\ 5 \end{pmatrix}$ and $b_j = 29 \times 2^j - 8 \times 3^j + 5 \times (-3)^j$

(d) $b_k \cong -13 \times 3^k$ for large $\underline{\text{odd}}$ k

$13 \times 3^k > 10^{10} \quad k > \dfrac{\log 10^{10} - \log 13}{\log 3} = \dfrac{10 - 1.11}{} = \dfrac{8.88}{0.477} = 18.6$

$\boxed{k \geq 19}$