Math1204 Handout 4: Polynomial Curve Fitting

Sometimes we may have data that we want to find out either an exact relationship between it or a best fit, in some sense. We can approach this method using Matrix Algebra as follows.

1. Given the m data points of the form $\{(x_1, y_1), \ldots (x_m, y_m)\}$, decide what kind of polynomial we want to fit to the data, say a polynomial of maximum degree n;

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n.$$

   - Ideally we would like an exact relationship so that we look for coefficients $a_0, \ldots, a_n$ such that $y_j = a_0 + a_1 x_j + a_2 x_j^2 + \cdots + a_n x_j^n$ for all values of j from 1 to m.
   - We form the $m \times (n+1)$ Vandermonde Matrix A from our x_j values:

$$A := \begin{pmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^n \\ 1 & x_2 & x_2^2 & \ldots & x_2^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_m & x_m^2 & \ldots & x_m^n \end{pmatrix}$$

   - If there is a solution to the equation $A\underline{v} = \underline{w} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$ then we have a polynomial which passes through all of the data points.

2. If there isn't an exact fit polynomial then we have to follow a slightly more complex procedure.
   - We still consider the equation $A\underline{v} = \underline{w}$, but this time pre-multiply both sides by A^T, giving a square matrix equation

$$(A^T A)\underline{v} = A^T \underline{w}$$

   - This can then be solved using row operations, inverses or the adjoint as appropriate.

Example: The best fit quadratic to the data: (-1,11), (1,0), (2,-1), (3,1), (0,-3)

$$A := \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 0 & 0 \end{pmatrix} \quad . \quad \underline{w} := \begin{pmatrix} 11 \\ 0 \\ -1 \\ 1 \\ -3 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 5 & 5 & 15 \\ 5 & 15 & 35 \\ 15 & 35 & 99 \end{pmatrix} \quad , \quad A^T \underline{w} = \begin{pmatrix} 8 \\ -10 \\ 16 \end{pmatrix} \quad , \quad \underline{v} = \begin{pmatrix} \frac{7}{5} \\ -\frac{29}{5} \\ 2 \end{pmatrix}$$

Thus the best fit polynomial is $f(x) = 2\,x^2 - \dfrac{29\,x}{5} + \dfrac{7}{5} = \dfrac{10x^2 - 29x + 7}{5}$.