Q1

$$M = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{pmatrix} \qquad y = \begin{pmatrix} -7 \\ 1 \\ 7 \\ 35 \end{pmatrix} \qquad M^TM = \begin{pmatrix} 16+1+1 & 8+1-1 & 4+1+1 \\ & & 4+1-1 \\ & & 1+1+1+1 \end{pmatrix} = \begin{pmatrix} 18 & 8 & 6 \\ 8 & 6 & 2 \\ 6 & 2 & 4 \end{pmatrix}$$

(a)

$$M^Ty = \begin{pmatrix} 1 & 0 & 1 & 4 \\ -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -7 \\ 1 \\ 7 \\ 35 \end{pmatrix} = \begin{pmatrix} -7+7+140 \\ 7+7+70 \\ -7+1+7+35 \end{pmatrix} = \begin{pmatrix} 140 \\ 84 \\ 36 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 18 & 8 & 6 & 140 \\ 8 & 6 & 2 & 84 \\ 6 & 2 & 4 & 36 \end{array}\right)$$

$R_1 \Leftarrow R_1 - 3R_2$ then $R_2 \times \frac{1}{2}$

$R_3 \Leftarrow R_3 - 2R_2$

$$\left(\begin{array}{ccc|c} -6 & -10 & 0 & -112 \\ 4 & 3 & 1 & 42 \\ -10 & -10 & 0 & -132 \end{array}\right)$$

$R_1 \Leftarrow \frac{R_1 - R_2}{4}$ $\quad R_3 \Leftarrow R_3 \times \frac{1}{10}$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 4 & 3 & 1 & 42 \\ 1 & 1 & 0 & \frac{132}{10} \end{array}\right)$$

$R_3 \Leftarrow R_3 - R_1$ $\quad R_2 \Leftarrow R_2 - 4R_1$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 3 & 1 & 22 \\ 0 & 1 & 0 & \frac{82}{10} \end{array}\right)$$

$R_2 \Leftarrow R_2 - 3R_3$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & -\frac{26}{10} \\ 0 & 1 & 0 & \frac{82}{10} \end{array}\right)$$

$a = 5 = \frac{25}{5}$ $\quad b = \frac{82}{10} = \frac{41}{5}$ $\quad c = -\frac{26}{10} = -\frac{13}{5}$

$$f(x) = \frac{25x^2 + 41x - 13}{5}$$

(b)

$$\left(\begin{array}{ccccc|c} 1 & -1 & 1 & -1 & 1 & -7 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 7 \\ 16 & 8 & 4 & 2 & 1 & 35 \end{array}\right)$$

$R_1 \Leftarrow \frac{R_1 - R_3}{-2}$

$R_3 \Leftarrow R_3 - R_2$

$R_4 \Leftarrow \frac{R_4 - R_2}{2}$

$$\left(\begin{array}{ccccc|c} 0 & 1 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 6 \\ 8 & 4 & 2 & 1 & 0 & 17 \end{array}\right)$$

$R_3 \Leftarrow R_3 - R_1$

$R_4 \Leftarrow R_4 - R_1$

$$\left(\begin{array}{ccccc|c} 0 & 1 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & -1 \\ 8 & 3 & 2 & 0 & 0 & 10 \end{array}\right)$$

$R_4 \Leftarrow \frac{R_4 - 2R_3}{3}$

$$\begin{array}{cccccc} p & q & r & s & t & \\ \end{array}$$
$$\left(\begin{array}{ccccc|c} 0 & 1 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & -1 \\ 2 & 1 & 0 & 0 & 0 & 4 \end{array}\right)$$

$R_1 \Leftarrow R_1 - R_4$

$p := k$

R1 $\quad -2k + s = 3$

R2 $\quad t = 1$

R3 $\quad r + k = -1$

R4 $\quad q + 2k = 4$

$$g(x) = (3+2k)x + 1 + (-1-k)x^2 + (4-2k)x^3 + kx^4$$
$$= k \times (x^4 - 2x^3 - x^2 + 2x) + 4x^3 - x^2 + 3x + 1$$
$$= k \times ((x-2)(x-1)x(x+1)) + 4x^3 - x^2 + 3x + 1$$

Q2

(b)

$$\left(\begin{array}{ccc|ccc} 6 & 4 & 1 & 1 & 0 & 0 \\ 7 & 5 & 2 & 0 & 1 & 0 \\ 5 & 3 & x & 0 & 0 & 1 \end{array}\right)$$

$R2 \Leftarrow R2 - R1$

$$\left(\begin{array}{ccc|ccc} 6 & 4 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 & 1 & 0 \\ 5 & 3 & x & 0 & 0 & 1 \end{array}\right)$$

$R1 \Leftarrow R1 - 4R2$
$R3 \Leftarrow R3 - 3R2$

$$\left(\begin{array}{ccc|ccc} 2 & 0 & -3 & 5 & -4 & 0 \\ 1 & 1 & 1 & -1 & 1 & 0 \\ 2 & 0 & x-3 & 3 & -3 & 1 \end{array}\right)$$

$R3 \Leftarrow R3 - R1$

$$\left(\begin{array}{ccc|ccc} 2 & 0 & -3 & 5 & -4 & 0 \\ 1 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & x & -2 & 1 & 1 \end{array}\right)$$

$R1 \Leftarrow R1/2 \quad R3 \Leftarrow R3/x$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -3/2 & 5/2 & -2 & 0 \\ 1 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2/x & 1/x & 1/x \end{array}\right)$$

$R2 \Leftarrow R2 - R1$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -3/2 & 5x & -4x & 0 \\ 0 & 1 & 5/2 & -7x & 6x & 0 \\ 0 & 0 & 1 & -4 & +2 & 2 \end{array}\right) \times \frac{1}{2x}$$

$R1 \Leftarrow R1 + \frac{3}{2} R3$
$R2 \Leftarrow R2 - \frac{5}{2} R3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 5x-6 & -4x+3 & 3 \\ 0 & 1 & 0 & 10-7x & 6x-5 & -5 \\ 0 & 0 & 1 & -4 & 2 & 2 \end{array}\right)$$
$$\overline{\quad 2x \quad}$$

(a)

$$\mathrm{Adj}(C) = \begin{pmatrix} \det\begin{pmatrix}5&2\\3&x\end{pmatrix} & -\det\begin{pmatrix}7&2\\5&x\end{pmatrix} & \det\begin{pmatrix}7&5\\5&3\end{pmatrix} \\ -\det\begin{pmatrix}4&1\\3&x\end{pmatrix} & \det\begin{pmatrix}6&1\\5&x\end{pmatrix} & -\det\begin{pmatrix}6&4\\5&3\end{pmatrix} \\ \det\begin{pmatrix}4&1\\5&2\end{pmatrix} & -\det\begin{pmatrix}6&1\\7&2\end{pmatrix} & \det\begin{pmatrix}6&4\\7&5\end{pmatrix} \end{pmatrix}^T$$

$$= \begin{pmatrix} 5x-6 & -(7x-10) & -4 \\ -(4x-3) & 6x-5 & +2 \\ 3 & -5 & 2 \end{pmatrix}^T$$

$$= \begin{pmatrix} 5x-6 & 3-4x & 3 \\ 10-7x & 6x-5 & -5 \\ -4 & +2 & 2 \end{pmatrix}$$

Check & find det(C)

$$C \times \mathrm{Adj}(C) = \begin{pmatrix} 30x-36+40-28x-4 & 18-24x+24x-20+2 & 0 \\ 35x-42+50-35x+8 & 21-28x+30x-25+4 & 0 \\ 25x-30+30-35x-4x & 15-20x+18x-15+2x & 2x \end{pmatrix}$$

$$= \begin{pmatrix} 2x & 0 & 0 \\ 0 & 2x & 0 \\ 0 & 0 & 2x \end{pmatrix}$$

so $C^{-1} = \dfrac{1}{2x} \begin{pmatrix} 5x-6 & 3-4x & 3 \\ 10-7x & 6x-5 & -5 \\ -4 & 2 & 2 \end{pmatrix}$

Q3

(a) $\left(\begin{array}{ccc|c} 5 & 9 & -7 & 0 \\ 2 & 2 & -3 & 0 \end{array}\right)$

$R_1 \leftarrow R_1 - 2R_2$

$\left(\begin{array}{ccc|c} 1 & 5 & -1 & 0 \\ 2 & 2 & -3 & 0 \end{array}\right)$

$R_2 \leftarrow R_2 - 2R_1$

$\left(\begin{array}{ccc|c} 1 & 5 & -1 & 0 \\ 0 & -8 & -1 & 0 \end{array}\right)$

$R_1 \leftarrow R_1 - R_2 \quad (R_2 \times -1)$

$\left(\begin{array}{ccc|c} 1 & 13 & 0 & 0 \\ 0 & 8 & 1 & 0 \end{array}\right)$

$\underline{n} = \begin{pmatrix} 13 \\ -1 \\ 8 \end{pmatrix} \qquad \underline{n} \cdot \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix} = -26 - 4 + 24 = -6$

Plane: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 13 \\ -1 \\ 8 \end{pmatrix} = -6$

(b) $\left(\begin{array}{ccc|c} 7 & -5 & 11 & 28 \\ 13 & -1 & 8 & -6 \end{array}\right)$

$R_1 \leftarrow R_1 - 5R_2$

$\left(\begin{array}{ccc|c} -58 & 0 & -29 & 58 \\ 13 & -1 & 8 & -6 \end{array}\right)$

$R_1 \leftarrow R_1 \times -\frac{1}{29} \quad R_2 \leftarrow R_2 \times -1$

$\left(\begin{array}{ccc|c} 2 & 0 & 1 & -2 \\ -13 & 1 & -8 & 6 \end{array}\right)$

$R_2 \leftarrow R_2 + 8R_1$

$\left(\begin{array}{ccc|c} 2 & 0 & 1 & -2 \\ 3 & 1 & 0 & -10 \end{array}\right)$

so $x = t \quad z = -2 - 2t \quad y = -10 - 3t \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} \times t + \begin{pmatrix} 0 \\ -10 \\ -2 \end{pmatrix}$

(c) Need K to have direction in P; eg $\begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$ from param

L ... Q; eg $\begin{pmatrix} 5 \\ 7 \\ 0 \end{pmatrix}$ from dot prod $\begin{pmatrix} 5 \\ 7 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ -5 \\ 11 \end{pmatrix} = 0$

They can meet at a point not in either plane eg $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ since $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ -5 \\ 11 \end{pmatrix} = 7 \neq 28$

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 13 \\ -1 \\ 8 \end{pmatrix} = 13 \neq -6$

So let K: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix} \times t$ and L: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 5 \\ 7 \\ 0 \end{pmatrix} \times s$

They meet when $t = 0$ and $s = 0$

not parallel to $\begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$ as directions are not multiples.

L and Q $\begin{pmatrix} 1+5s \\ 7s \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ -5 \\ 11 \end{pmatrix} = 7 \neq 28$

K and P $\begin{pmatrix} 1+2t \\ 2t \\ -3t \end{pmatrix} \cdot \begin{pmatrix} 13 \\ -1 \\ 8 \end{pmatrix} = 13 + 26t - 2t - 24t = 13 \neq -6$

(a) $M = \begin{pmatrix} -\frac{375}{2} & 459 \\ -77 & \frac{377}{2} \end{pmatrix}$ $\begin{pmatrix} e_k \\ f_k \end{pmatrix} = M^k \begin{pmatrix} 297 \\ 231 \end{pmatrix}$

$$\det(M - \lambda I) = \left(-\frac{375}{2} - \lambda\right)\left(\frac{377}{2} - \lambda\right) - -77 \times 459$$
$$= \lambda^2 + \left(\frac{375-377}{2}\right)\lambda - \frac{141375}{4} + 35343$$
$$= \lambda^2 - \lambda - \frac{3}{4} = \left(\lambda - \frac{3}{2}\right)\left(\lambda + \frac{1}{2}\right) = \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2}\lambda - \frac{3}{4}$$

so $\lambda_1 = 3/2$ $\lambda_2 = -1/2$

$\underline{V}_1$: $M - \lambda_1 I = \begin{pmatrix} -\frac{378}{2} & 459 \\ -77 & \frac{374}{2} \end{pmatrix} = \begin{pmatrix} -189 & 459 \\ -77 & 187 \end{pmatrix}$ $\left(\begin{array}{cc|c} -189 & 459 & 0 \\ -77 & 187 & 0 \end{array}\right)$ $R_1 \leftarrow R_1 - \frac{189}{77} R_2$

so $\underline{V}_1 = \begin{pmatrix} 187 \\ 77 \end{pmatrix}$ or $\underline{\begin{pmatrix} 17 \\ 7 \end{pmatrix}}$ $\left(\begin{array}{cc|c} 0 & 0 & 0 \\ -77 & 187 & 0 \end{array}\right)$

$\underline{V}_2$: $M - \lambda_2 I = \begin{pmatrix} -187 & 459 \\ -77 & 189 \end{pmatrix}$ $\left(\begin{array}{cc|c} -187 & 459 & 0 \\ -77 & 189 & 0 \end{array}\right)$ $R_1 \leftarrow R_1 - \frac{187}{77} R_2$ $\left(\begin{array}{cc|c} 0 & 0 & 0 \\ -77 & 189 & 0 \end{array}\right)$

$\underline{V}_2 = \begin{pmatrix} 189 \\ 77 \end{pmatrix}$ or $\underline{\begin{pmatrix} 27 \\ 11 \end{pmatrix}}$

(b) $P = \begin{pmatrix} 17 & 27 \\ 7 & 11 \end{pmatrix}$ $P^{-1} = \frac{1}{187-189} \begin{pmatrix} 11 & -27 \\ -7 & 17 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -11 & 27 \\ 7 & -17 \end{pmatrix}$ $D = \begin{pmatrix} 3/2 & 0 \\ 0 & -1/2 \end{pmatrix}$

Thus $M^k \begin{pmatrix} 297 \\ 231 \end{pmatrix} = \begin{pmatrix} 17 & 27 \\ 7 & 11 \end{pmatrix} \begin{pmatrix} (3/2)^k & 0 \\ 0 & (-1/2)^k \end{pmatrix} \frac{1}{2} \begin{pmatrix} -11 & 27 \\ 7 & -17 \end{pmatrix} \begin{pmatrix} 297 \\ 231 \end{pmatrix}$

$$= \begin{pmatrix} 17 \times \left(\frac{3}{2}\right)^k & 27 \times \left(-\frac{1}{2}\right)^k \\ 7 \times \left(\frac{3}{2}\right)^k & 11 \times \left(-\frac{1}{2}\right)^k \end{pmatrix} \begin{pmatrix} 1485 \\ -1848 \end{pmatrix}$$

$$\begin{pmatrix} e_k \\ f_k \end{pmatrix} = \begin{pmatrix} 25245 \times \left(\frac{3}{2}\right)^k - 49896 \times \left(\frac{-1}{2}\right)^k \\ 10395 \times \left(\frac{3}{2}\right)^k - 20328 \times \left(-\frac{1}{2}\right)^k \end{pmatrix}$$

(c) $\frac{e_0}{231} = \frac{27}{11}$ so $e_0 = 21 \times 27 = 567$

If $e_0 > 567$ then the coeff of $\left(\frac{3}{2}\right)^k$ will be $\frac{1}{2} \times (231 \times 27 - 11 \times e_0) < 0$

Q5 $\quad \sum \alpha_i \underline{w}_i = \underline{0}$

(a)
$$\left(\begin{array}{cccc|c} -1 & 1 & 5 & 3 & 0 \\ 4 & 4 & 2 & 3 & 0 \\ 1 & 1 & -3 & -1 & 0 \\ 1 & 2 & 2 & 2 & 0 \end{array}\right)$$

$R_1 \Leftarrow R_1 + R_3$
$R_2 \Leftarrow R_2 - 4R_3$
$R_4 \Leftarrow R_4 - R_3$

$$\left(\begin{array}{cccc|c} 0 & 2 & 2 & 2 & 0 \\ 0 & 0 & 14 & 7 & 0 \\ 1 & 1 & -3 & -1 & 0 \\ 0 & 1 & 5 & 3 & 0 \end{array}\right)$$

$R_1 \Leftarrow \frac{R_1}{2} - R_4 \quad R_2 \Leftarrow R_2 \times \frac{1}{7}$
$R_3 \Leftarrow R_3 - R_4$

$$\left(\begin{array}{cccc|c} 0 & 0 & -4 & -2 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 1 & 0 & -8 & -4 & 0 \\ 0 & 1 & 5 & 3 & 0 \end{array}\right)$$

$R_1 \Leftarrow R_1 + 2R_2$
$R_3 \Leftarrow R_3 + 4R_2$
$R_4 \Leftarrow R_4 - 3R_2$

$$\left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{array}\right)$$

Thus $\alpha_1 = 0$
$\alpha_4 = -2\alpha_3$
$\alpha_2 = \alpha_3$

and

$\underline{w}_2 + \underline{w}_3 - 2\underline{w}_4$

$$= \begin{pmatrix} 1+5-6 \\ 4+2-6 \\ 1-3+2 \\ 2+2-4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

(b) we need $\underline{n} \cdot \underline{w}_1 = \underline{n} \cdot \underline{w}_2 = \underline{n} \cdot \underline{w}_4 = 0$

$$\left(\begin{array}{cccc|c} -1 & 4 & 1 & 1 & 0 \\ 1 & 4 & 1 & 2 & 0 \\ 3 & 3 & -1 & 2 & 0 \end{array}\right)$$

$R_2 \Leftarrow R_2 - R_1 \quad R_3 \Leftarrow R_3 + R_1$

$$\left(\begin{array}{cccc|c} -1 & 4 & 1 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 2 & 7 & 0 & 3 & 0 \end{array}\right)$$

$R_1 \Leftarrow R_1 - R_2 \quad R_3 \Leftarrow R_3 - 3R_2$

$$\left(\begin{array}{cccc|c} -3 & 4 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ -4 & 7 & 0 & 0 & 0 \end{array}\right)$$

$R_1 \Leftarrow R_1 - \frac{4}{7}R_3 \quad R_3 \Leftarrow R_3 \times \frac{1}{7} \qquad -3 + \frac{16}{7} = \frac{5}{7}$

$$\left(\begin{array}{cccc|c} 5/7 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ -\frac{4}{7} & 1 & 0 & 0 & 0 \end{array}\right) \qquad \underline{n} = \begin{pmatrix} 7 \\ +4 \\ +5 \\ -14 \end{pmatrix}$$

so $\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 7 \\ 4 \\ 5 \\ -14 \end{pmatrix} = 0$ is the hyperplane

$\underline{w}_5 \cdot \begin{pmatrix} 7 \\ 4 \\ 5 \\ -14 \end{pmatrix} = 14 + 4 + 10 - 28 = 0$ ✓

(c) $\begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}$ are indep since:

$C_2 \Leftarrow C_2 - C_1 \quad C_3 \Leftarrow C_3 - C_1$

$$\det = \det\begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 2 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & 1 \\ 0 & 2 & 3 \end{pmatrix} = \det\begin{pmatrix} 1 & 0 & 0 \\ -1 & 3 & 2 \\ 0 & 2 & 3 \end{pmatrix}$$

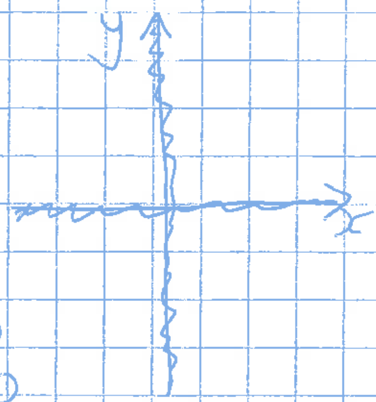
$R_3 \Leftarrow R_3 + R_4$

$= 9 - 4 = 5 \neq 0$

$C_1 \Leftarrow C_1 - C_2 \quad C_4 \Leftarrow C_4 - C_3$

Q6

(a)



$xy=0$

$\Rightarrow x=0$

or $y=0$

A1 false since $(2,0)$ has $y=0$

$+\ (0,3)$ has $x=0$

$(2,3)$ has neither $=0$

A4 $(0,0)$ is in S since $x=0$ and $y=0$

M1 if (x,y) is in S then $xy=0$

But $(\alpha x, \alpha y)$ has $x'y' = (\alpha x)(\alpha y) = \alpha^2 xy$

$= \alpha^2 0$

$= 0$

so M1 true

(b) $2(A+X)B = A(B+I) = AB + AI = AB + A$ Distrib/Ident

Multiply by B^{-1} on right

LHS $2(A+X)BB^{-1} = 2(A+X)I = (AB+A)B^{-1} = ABB^{-1} + AB^{-1}$ Inverse Distrib

$= 2(A+X)$ $\quad = A + AB^{-1}$

Scalar distrib

So $2A + 2X = A + AB^{-1}$

$2X = A - 2A + AB^{-1} = AB^{-1} - A = A(B^{-1} - I)$

$X = \frac{1}{2} A(B^{-1} - I)$

If $X=0$ then $A=0$, $B^{-1} = I$ or $B^{-1} = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$ so $B = \frac{1}{4}\begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}$

then $B^{-1} - I = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$

and if $A = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$ then $A(B^{-1} - I)$

$= \begin{pmatrix} 2-2 & 2-2 \\ -4+4 & -4+4 \end{pmatrix}$

Q7

(a) $A\underline{v} = \begin{pmatrix} 6 & 24 & 36 \\ -2 & -10 & -18 \\ 1 & 6 & 11 \end{pmatrix}\begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 24-48+36 \\ -8+20-18 \\ 4-12+11 \end{pmatrix} = \begin{pmatrix} 12 \\ -6 \\ 3 \end{pmatrix} = 3\times\begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$

so it is an eigenvector and $\lambda_1 = 3$

(b) $\det(A-\lambda I) = \det\left(\begin{pmatrix} 6-\lambda & 24 & 36 \\ -2 & -10-\lambda & -18 \\ 1 & 6 & 11-\lambda \end{pmatrix}\right) \underset{\substack{\text{Row2} \\ \leftarrow R2+2R3}}{=} \det\left(\begin{pmatrix} 6-\lambda & 24 & 36 \\ 0 & 2-\lambda & 4-2\lambda \\ 1 & 6 & 11-\lambda \end{pmatrix}\right)$

$\underset{\substack{\text{col3}\leftarrow \\ c3-2c2}}{=} \det\left(\begin{pmatrix} 6-\lambda & 24 & -12 \\ 0 & 2-\lambda & 0 \\ 1 & 6 & -1-\lambda \end{pmatrix}\right)$

$= (2-\lambda)\times(\lambda^2-6\lambda+\lambda+12-6)$

$= (2-\lambda)\times(\lambda^2-5\lambda+6) = (2-\lambda)(\lambda-3)(\lambda+2)$

Thus $\lambda_2 = \lambda_3 = 2$

and $A-2I$: $\left(\begin{array}{ccc|c} 4 & 24 & 36 & 0 \\ -2 & -12 & -18 & 0 \\ 1 & 6 & 9 & 0 \end{array}\right)$

$R_1 \leftarrow R_1 - 4R_3 \quad R_2 \leftarrow R_2 + 2R_3$

$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 6 & 9 & 0 \end{array}\right)$

So $y = k \quad z = l \quad x = -6k - 9l$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -6 \\ 1 \\ 0 \end{pmatrix}\times k + \begin{pmatrix} -9 \\ 0 \\ 1 \end{pmatrix}\times l$

(c)

$\left(\begin{array}{ccc|c} -6 & 1 & 0 & 0 \\ -9 & 0 & 1 & 0 \end{array}\right)$

gives $\underline{n} = \begin{pmatrix} 1 \\ 6 \\ 9 \end{pmatrix}$

but $A\begin{pmatrix} 1 \\ 6 \\ 9 \end{pmatrix} = \begin{pmatrix} 6+144+325 \\ -2-60-162 \\ 1+36+99 \end{pmatrix} = \begin{pmatrix} 475 \\ -224 \\ 136 \end{pmatrix}$

since one term is negative, it isn't an eigenvector

G-S $\Rightarrow \begin{pmatrix} -9 \\ 0 \\ 1 \end{pmatrix} - \dfrac{\begin{pmatrix} -6 \\ 1 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} -9 \\ 0 \\ 1 \end{pmatrix}}{\begin{pmatrix} -6 \\ 1 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} -6 \\ 1 \\ 0 \end{pmatrix}}\begin{pmatrix} -6 \\ 1 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} -9 \\ 0 \\ 1 \end{pmatrix} - \dfrac{54}{37}\begin{pmatrix} -6 \\ 1 \\ 0 \end{pmatrix}$

$= \dfrac{1}{37}\begin{pmatrix} 6\times54-9\times37 \\ -54 \\ 37 \end{pmatrix} = \dfrac{1}{37}\begin{pmatrix} -9 \\ -54 \\ 37 \end{pmatrix}$

$A\begin{pmatrix} -9 \\ -54 \\ 37 \end{pmatrix} = \begin{pmatrix} -18 \\ -108 \\ 74 \end{pmatrix}$ so is eigenvector of value 2