Math1204 Handout 3: Eigenvalues and vectors

Given any $n \times n$ matrix M, we can find their eigenvalues and eigenvectors as follows:

1. Find the determinant of the matrix $(M - \lambda I)$ (λ is lambda, a greek letter used for eigenvalues).
   - You should get a polynomial of degree n, the *characteristic polynomial* $p(\lambda)$.
   - The eigenvalues of M are the roots of $p(\lambda) = 0$, you can find them using factorisation or other methods.
   - Usually we call these eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ in the order we find them.
2. For each eigenvalue λ_j, we can find an eigenvector, $\underline{v}_j$ (an $n \times 1$ matrix is called a vector and is identified as such by underlining it; others use $\mathbf{v}_j$, $\vec{v}_j$ or $\overrightarrow{v_j}$).
   - We are solving the matrix equation $M\underline{v}_j = \lambda_j \underline{v}_j$ which is equivalent to the homogeneous equation $(M - \lambda_j I)\underline{v}_j = \underline{0}$, where $\underline{0}$ is the $n \times 1$ all zeros matrix.
   - Subtract the value of λ_j from the diagonal entries of M and augment a column of zeros, then pivot as many times as possible.
   - The solution(s) (without any letters multiplying them) are the eigenvector(s) of M corresponding to eigenvalue λ_j.
   - We can check by multiplying M by each vector $\underline{v}_j$ and should get λ_j times the vector, since $M\underline{v}_j = \lambda_j \underline{v}_j$.
3. Each eigenvalue will give at least one row of zeros after row operations. If λ_j appears more than once as a root, it can give as many as that number of eigenvectors, but not necessarily more than one.

Example: If $M := \begin{pmatrix} -13 & 24 & 4 \\ -3 & 5 & 1 \\ -15 & 30 & 4 \end{pmatrix}$ then we take the determinant of $M - \lambda I$ which can be calculated to be $\lambda^3 + 4\lambda^2 + 5\lambda + 2 = (\lambda + 2)(\lambda + 1)^2$ (all combinations of row/col ops work for this $M - \lambda I$). Thus the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = -1$, and λ_2 has multiplicity 2. We can now use row operations to solve $(M - (-2)I)\underline{v} = \underline{0}$ and $(M - (-1)I)\underline{v} = \underline{0}$, that is, after using row operations we get these matrices to solve for $\underline{v}_1$ and $\underline{v}_2$:

$$\left(\begin{array}{ccc:c} -11 & 24 & 4 & 0 \\ -3 & 7 & 1 & 0 \\ -15 & 30 & 6 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc:c} 1 & -4 & 0 & 0 \\ 0 & -5 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \quad \left(\begin{array}{ccc:c} -12 & 24 & 4 & 0 \\ -3 & 6 & 1 & 0 \\ -15 & 30 & 5 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc:c} -3 & 6 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

From this we can get that the eigenvectors of M are $\begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix}$.