Math1204 Handout 4: Polynomial Curve Fitting

Sometimes we may have data that we want to find out either an exact relationship between it or a best fit, in some sense. We can approach this method using Matrix Algebra as follows.

1. Given the m data points of the form $\{(x_1, y_1), \ldots (x_m, y_m)\}$, decide what kind of polynomial we want to fit to the data, say a polynomial of maximum degree n;

$$f(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0$$

   - Ideally we would like an exact relationship so that we look for coefficients $a_0, \ldots, a_n$ such that $y_j = a_n x_j^n + \cdots + a_2 x_j^2 + a_1 x_j + a_0$ for all values of j from 1 to m.
   - We form the $m \times (n+1)$ Matrix M from our x_j and $\underline{y}$ from the y_j.

$$M := \begin{pmatrix} x_1^n & \ldots & x_1^2 & x_1 & 1 \\ x_2^n & \ldots & x_2^2 & x_2 & 1 \\ \vdots & & \vdots & \vdots & \vdots \\ x_m^n & \ldots & x_m^2 & x_m & 1 \end{pmatrix}, \quad \underline{y} := \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$

   - If there is a solution to the matrix equation $M\underline{v} = \underline{y}$ then we have a polynomial which passes through all of the data points.

2. If there isn't an exact fit polynomial then we have to follow a slightly more complex procedure.
   - We still consider the equation $M\underline{v} = \underline{y}$, but this time left multiply both sides by M^T, giving an $(n+1) \times (n+1)$ matrix equation: $(M^T M)\underline{v} = (M^T \underline{y})$
   - This can then be solved using row operations, inverses or the adjoint as appropriate.

Example: The best fit quadratic to the data: (2,2), (-1,4), (4,3), (1,-3)

$$M := \begin{pmatrix} 4 & 2 & 1 \\ 1 & -1 & 1 \\ 16 & 4 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \underline{y} := \begin{pmatrix} 2 \\ 4 \\ 3 \\ -3 \end{pmatrix}, \quad \begin{array}{l} \sum_{j=1}^{5} x_j^0 = 4, \sum_{j=1}^{4} x_j = 2 - 1 + 4 + 1 = 6 \\ \sum_{j=1}^{5} x_j^2 = 22, \sum_{j=1}^{5} x_j^3 = 72, \sum_{j=1}^{5} x_j^4 = 274 \end{array}$$

Using the sums of powers of the x_j and combining them with the y_j by multiplication:

$$M^T M = \begin{pmatrix} 274 & 72 & 22 \\ 72 & 22 & 6 \\ 22 & 6 & 4 \end{pmatrix}, \quad M^T \underline{y} = \begin{pmatrix} 57 \\ 9 \\ 6 \end{pmatrix} \text{ and row ops give } \quad \underline{v} = \begin{pmatrix} \frac{2}{3} \\ -2 \\ \frac{5}{6} \end{pmatrix}$$

Thus the best fit quadratic polynomial is $f(x) = \dfrac{2}{3}x^2 - 2\,x + \dfrac{5}{6} = \dfrac{4x^2 - 12x + 5}{6}$.

Tabulating we see:

x_j	y_j	$f(x_j)$	$y_j - f(x_j)$
2	2	$-1/2$	$5/2$
-1	4	$7/2$	$1/2$
4	3	$7/2$	$-1/2$
1	-3	$-1/2$	$-5/2$

and note that the sum of the differences is 0.