21 (a)

$$\operatorname{adj}(M) = \begin{pmatrix} \det\begin{pmatrix}-3 & 2\\ x & -5\end{pmatrix} & -\det\begin{pmatrix}6 & 2\\ 4 & -5\end{pmatrix} & +\det\begin{pmatrix}6 & -3\\ 4 & x\end{pmatrix} \\ -\det\begin{pmatrix}3 & 1\\ x & -5\end{pmatrix} & \det\begin{pmatrix}3 & 1\\ 4 & -5\end{pmatrix} & -\det\begin{pmatrix}3 & 3\\ 4 & x\end{pmatrix} \\ \det\begin{pmatrix}3 & 1\\ -3 & 2\end{pmatrix} & -\det\begin{pmatrix}3 & 1\\ 6 & 2\end{pmatrix} & \det\begin{pmatrix}3 & 3\\ 6 & -3\end{pmatrix} \end{pmatrix}^T$$

$$= \begin{pmatrix} 15-2x & -(-30-8) & 6x+12 \\ -(-15-x) & -15-4 & -(3x-12) \\ 6--3 & -(6-6) & -9-18 \end{pmatrix}^T$$

$$= \begin{pmatrix} 15-2x & 38 & 6x+12 \\ 15+x & -19 & -3x+12 \\ 9 & 0 & -27 \end{pmatrix}^T$$

$$= \begin{pmatrix} 15-2x & 15+x & 9 \\ 38 & -19 & 0 \\ 6x+12 & -3x+12 & -27 \end{pmatrix}$$

(b)

$$M \times \operatorname{adj}(M) = \begin{pmatrix} 45-6x+114+6x+12 & 45+3x-57-3x+12 & 27+0-27 \\ 90-12x-114+12x+24 & 90+6x+57-6x+24 & 54+0-24 \\ 60-8x+38x-30x-60 & 60+4x-19x+15x-60 & 36+135 \end{pmatrix}$$

$$= \begin{pmatrix} 171 & 0 & 0 \\ 0 & 171 & 0 \\ 0 & 0 & 171 \end{pmatrix}$$

$$= 171\, I \qquad \text{so } \det(M) = 171$$

(c)

$$\det(N) = 3\times\det\begin{pmatrix}-3 & 2\\ x & -5\end{pmatrix} - y\times\det\begin{pmatrix}6 & 2\\ 4 & -5\end{pmatrix} + 1\times\det\begin{pmatrix}6 & -3\\ 4 & x\end{pmatrix}$$

$$= 3\times(15-2x) - y\times(-38) + (6x+12)$$

$$= 45-6x+38y+6x+12 = 57+38y$$

N is singular when $\det(N)=0$ so $57+38y=0$ $\quad y = \frac{-57}{38} = \frac{-3}{2}$.

Q2

$$\det(B-\lambda I) = \frac{1}{4}\det\left(\begin{pmatrix} -24-2\lambda & 35 \\ -15 & 22-2\lambda \end{pmatrix}\right)$$

$$= \frac{1}{4}\left(4\lambda^2 - 44\lambda + 48\lambda - 528 + 15\times 35\right)$$

$$= \lambda^2 + \lambda - \frac{3}{4}$$

$$\lambda = \frac{-1 \pm \sqrt{1 - 4\times 1\times -\frac{3}{4}}}{2} = \frac{-1 \pm \sqrt{4}}{2} = \frac{-1\pm 2}{2} = \frac{1}{2} \text{ or } \frac{-3}{2}$$

$$\det(B-\lambda I) = \left(\lambda - \frac{1}{2}\right)\left(\lambda + \frac{3}{2}\right)$$

$\lambda_1 = \frac{1}{2}$ $\quad \lambda_2 = \frac{-3}{2}$ $\quad |\lambda_2| = \frac{3}{2} > \frac{1}{2} = |\lambda_1|$ so λ_2 is dominant

$\underline{v}_1$: $2\lambda_1 = 1$

$$\left(\begin{array}{cc|c} -25 & 35 & 0 \\ -15 & 21 & 0 \end{array}\right) \quad \begin{array}{l} R_1 \leftarrow R_1 \div 5 \\ R_2 \leftarrow R_2 \div 3 \end{array} \quad \left(\begin{array}{cc|c} -5 & 7 & 0 \\ -5 & 7 & 0 \end{array}\right) \quad R_2 \leftarrow R_2 - R_1 \quad \underline{v}_1 = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

$$B\underline{v}_1 = \frac{1}{2}\begin{pmatrix} -24 & 35 \\ -15 & 22 \end{pmatrix}\begin{pmatrix} 7 \\ 5 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -168+175 \\ -105+110 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 7 \\ 5 \end{pmatrix} = \frac{1}{2}\underline{v}_1$$

$\underline{v}_2$: $2\lambda_2 = -3$

$$\left(\begin{array}{cc|c} -21 & 35 & 0 \\ -15 & 25 & 0 \end{array}\right) \quad R_1 \leftarrow R_1 - \frac{7}{5}R_2 \quad \left(\begin{array}{cc|c} 0 & 0 & 0 \\ -15 & 25 & 0 \end{array}\right) \quad \underline{v}_2 = \begin{pmatrix} 25 \\ 15 \end{pmatrix} \text{ or } \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

$$B\underline{v}_2 = \frac{1}{2}\begin{pmatrix} -24 & 35 \\ -15 & 22 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -120+105 \\ -75+66 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -15 \\ -9 \end{pmatrix} = -\frac{3}{2}\underline{v}_2$$

$$P = \begin{pmatrix} 7 & 5 \\ 5 & 3 \end{pmatrix} \quad P^{-1} = \frac{1}{21-25}\begin{pmatrix} 3 & -5 \\ -5 & 7 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -3 & 5 \\ 5 & -7 \end{pmatrix}$$

$$B^k = PD^kP^{-1} = \begin{pmatrix} 7 & 5 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} \left(\frac{1}{2}\right)^k & 0 \\ 0 & \left(-\frac{3}{2}\right)^k \end{pmatrix}\frac{1}{4}\begin{pmatrix} -3 & 5 \\ 5 & -7 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 7\times\left(\frac{1}{2}\right)^k & 5\times\left(-\frac{3}{2}\right)^k \\ 5\times\left(\frac{1}{2}\right)^k & 3\times\left(-\frac{3}{2}\right)^k \end{pmatrix}\begin{pmatrix} -3 & 5 \\ 5 & -7 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} -21\times\left(\frac{1}{2}\right)^k + 25\times\left(-\frac{3}{2}\right)^k & 35\times\left(\frac{1}{2}\right)^k - 35\times\left(-\frac{3}{2}\right)^k \\ -15\times\left(\frac{1}{2}\right)^k + 15\times\left(-\frac{3}{2}\right)^k & 25\times\left(\frac{1}{2}\right)^k - 21\times\left(-\frac{3}{2}\right)^k \end{pmatrix}$$

Need $\begin{pmatrix} 7 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} \alpha \\ 75 \end{pmatrix}$ to be the same ratio

$$\alpha = 75\times\frac{7}{5} = 15\times 7 = \underline{\underline{105}}$$

Q3

$$\left(\begin{array}{ccccc|c} 0 & 7 & 4 & 3 & -1 & 2 \\ 3 & 5 & 5 & 3 & 1 & 1 \\ 5 & 6 & 7 & 4 & 2 & 1 \\ 2 & 1 & 2 & 1 & 1 & 0 \end{array}\right)$$

$R_2 \Leftarrow R_2 + R_1 \quad R_3 \Leftarrow R_3 + 2R_1 \quad R_4 \Leftarrow R_4 + R_1$

$$\left(\begin{array}{ccccc|c} 0 & 7 & 4 & 3 & -1 & 2 \\ 3 & 12 & 9 & 6 & 0 & 3 \\ 5 & 20 & 15 & 10 & 0 & 5 \\ 2 & 8 & 6 & 4 & 0 & 2 \end{array}\right)$$

$R_2 \Leftarrow R_2 \times \frac{1}{3} \quad R_3 \Leftarrow R_3 \times \frac{1}{5} \quad R_4 \Leftarrow R_4 \times \frac{1}{2}$

$$\left(\begin{array}{ccccc|c} 0 & 7 & 4 & 3 & -1 & 2 \\ 1 & 4 & 3 & 2 & 0 & 1 \\ 1 & 4 & 3 & 2 & 0 & 1 \\ 1 & 4 & 3 & 2 & 0 & 1 \end{array}\right)$$

$R_3 \Leftarrow R_3 - R_2 \quad R_4 \Leftarrow R_4 - R_2$

$$\left(\begin{array}{ccccc|c} 0 & 7 & 4 & 3 & -1 & 2 \\ 1 & 4 & 3 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$7w + 4x + 3y - z = 2$

$v + 4w + 3x + 2y = 1$

$w = r, \; x = s, \; y = t$

$$\begin{pmatrix} v \\ w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 7 \end{pmatrix} \times r + \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 4 \end{pmatrix} \times s + \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 3 \end{pmatrix} \times t + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -2 \end{pmatrix}$$

Rank is 2 because there are two pivots possible before all zeros left

Rank 1 all rows are multiples. eg

$$\left(\begin{array}{ccccc|c} 1 & 2 & 3 & 5 & 8 & 13 \\ 2 & 4 & 6 & 10 & 16 & 26 \\ -1 & -2 & -3 & -5 & -8 & -13 \\ 10 & 20 & 30 & 50 & 80 & 130 \end{array}\right)$$

$$\begin{pmatrix} 1 \\ 2 \\ -1 \\ 10 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 5 & 8 \end{pmatrix} = 13 \begin{pmatrix} 1 \\ 2 \\ -1 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -1 & 3 \\ -2 & 5 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 7 & 4 & 3 & -1 \\ 1 & 4 & 3 & 2 & 0 \end{pmatrix}$$

Q4

(a) $F\begin{pmatrix}9\\1\\10\end{pmatrix} = \begin{pmatrix}-99 & -20 & +200\\ 270 & +39 & -300\\ 135 & +15 & -60\end{pmatrix} = \begin{pmatrix}81\\9\\90\end{pmatrix} = 9\times\begin{pmatrix}9\\1\\10\end{pmatrix}$

$F^T\begin{pmatrix}9\\1\\10\end{pmatrix} = \begin{pmatrix}-99 & +30 & +150\\ -180 & +39 & +150\\ 180 & -30 & \cancel{+}60\end{pmatrix} = \begin{pmatrix}81\\9\\90\end{pmatrix} = 9\times\begin{pmatrix}9\\1\\10\end{pmatrix}$

$(F-9I)\underline{v}=\underline{0}$ $\left(\begin{array}{ccc|c}-20 & -20 & 20 & 0\\ 30 & 30 & -30 & 0\\ 15 & 15 & -15 & 0\end{array}\right)$ $R_1 \leftarrow R_1\times\frac{1}{20}$ $R_2\leftarrow R_2\times\frac{1}{30}$ $R_3\leftarrow R_3\times\frac{1}{15}$

$\left(\begin{array}{ccc|c}-1 & -1 & 1 & 0\\ -1 & -1 & 1 & 0\\ -1 & -1 & 1 & 0\end{array}\right)$ $R_2\leftarrow R_2-R_1$ $R_3\leftarrow R_3-R_1$

$-x-y+z=0$ $z=x+y$
$x=s$ $y=t$ $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix}s + \begin{pmatrix}0\\1\\1\end{pmatrix}t$

$(F^T-9I)\underline{w}=\underline{0}$ $\left(\begin{array}{ccc|c}-20 & 30 & 15 & 0\\ -20 & 30 & 15 & 0\\ 20 & -30 & -15 & 0\end{array}\right)$ $R_2\leftarrow R_2-R_1$ $R_3\leftarrow R_3+R_1$ $\left(\begin{array}{ccc|c}-20 & 30 & 15 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right)$

$-20x+30y+15z=0$ $z=\frac{20}{15}x-\frac{30}{15}y = \frac{4}{3}x-2y$

$x=k$ $y=l$ $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\0\\4/3\end{pmatrix}\times k + \begin{pmatrix}0\\1\\-2\end{pmatrix}\times l$

$\begin{pmatrix}9\\1\\10\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix}\times 9 + \begin{pmatrix}0\\1\\1\end{pmatrix}\times 1 = \begin{pmatrix}9\\1\\10\end{pmatrix}$ $\begin{pmatrix}9\\1\\10\end{pmatrix} = \begin{pmatrix}1\\0\\4/3\end{pmatrix}\times 9 + \begin{pmatrix}0\\1\\-2\end{pmatrix}\times 1 = \begin{pmatrix}9\\1\\12-2\end{pmatrix} = \begin{pmatrix}9\\1\\10\end{pmatrix}$