Q1(a)

$$\det(A) = \det\left(\begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -1 & 2 \\ 2 & -2 & 1 & y \\ 0 & x & 0 & -2 \end{pmatrix}^{\text{transpose}}\right) \underset{R_3 \leftarrow R_3 - 2R_1}{=} \det\left(\begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & -6 & 1 & y-4 \\ 0 & x & 0 & -2 \end{pmatrix}\right)$$

$$= 1 \times \det\left(\begin{pmatrix} 1 & -1 & 2 \\ -6 & 1 & y-4 \\ x & 0 & -2 \end{pmatrix}\right) \underset{R_2 \leftarrow R_2 + R_1}{=} \det\left(\begin{pmatrix} 1 & -1 & 2 \\ -5 & 0 & y-2 \\ x & 0 & -2 \end{pmatrix}\right)$$

$$= -1 \times -1 \times \det\left(\begin{pmatrix} -5 & y-2 \\ x & -2 \end{pmatrix}\right)$$

$$= +10 - x(y-2)$$

$$= +10 - xy + 2x$$

(b) $10 - xy + 2x = 0$ when $x(y-2) = 10$

or $x = \dfrac{10}{y-2}$ if $y \neq 2$

thus if $y = 2$ $\det(A) = 10 - 2x + 2x = 10 \neq 0$

so we need $y-2$ to be -10 -5 -2 -1 1 2 5 10

so $y =$ ~~-12 -7 -4 -3 -1 0 3 8~~

-8 -3 0 1 3 4 7 12

$x = \dfrac{10}{y-2} = -1$ -2 -5 -10 10 5 2 1

Q2(a) $\det(B-\lambda I) = \det\left(\begin{pmatrix}13-\lambda & 12 & -6\\ 27 & 13-\lambda & -9 \\ 99 & 66 & -38-\lambda\end{pmatrix}\right)$

$R_2 \Leftarrow R_2 - \frac{3}{2}R_1$

$= \det\left(\begin{pmatrix}13-\lambda & 12 & -6\\ \frac{15+3\lambda}{2} & -5-\lambda & 0\\ 99 & 66 & -38-\lambda\end{pmatrix}\right)$

$27-\frac{3}{2}(13-\lambda)$

$=\frac{54-39}{2}+\frac{3}{2}\lambda$

$=\frac{15+3\lambda}{2}$

Transpose

$= \det\left(\begin{pmatrix}13-\lambda & \frac{15+3\lambda}{2} & 99\\ 12 & -5-\lambda & 66\\ -6 & 0 & -38-\lambda\end{pmatrix}\right)$

$R_1 \Leftarrow R_1 + \frac{3}{2}R_2$

$= \det\left(\begin{pmatrix}31-\lambda & 0 & 198\\ 12 & -5-\lambda & 66\\ -6 & 0 & -38-\lambda\end{pmatrix}\right)$

$=(-5-\lambda)\times\left((31-\lambda)(-38-\lambda) - -6\times198\right)$

$=(-5-\lambda)\times\left(\lambda^2+7\lambda-31\times38+6\times198\right)$

$=(-5-\lambda)\times\left(\lambda^2+7\lambda+10\right)$

$=-(\lambda+5)\times(\lambda+2)(\lambda+5)$

eigenvalues are $-5, -5$ and -2

~~det~~ $(B+5I \mid 0)$ $\left(\begin{array}{ccc|c}18 & 12 & -6 & 0\\ 27 & 18 & -9 & 0\\ 99 & 66 & -33 & 0\end{array}\right)$ $R_1 \Leftarrow R_1\times\frac{-1}{6}$, $R_2 \Leftarrow R_2 + 9\,\text{new}R_1$, $R_3 \Leftarrow R_3 + 33\,\text{new}R_1$ $\left(\begin{array}{ccc|c}-3 & -2 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right)$

Thus $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x\\y\\3x+2y\end{pmatrix} = \begin{pmatrix}1\\0\\3\end{pmatrix}\times k + \begin{pmatrix}0\\1\\2\end{pmatrix}\times l$

and when $k=1$, $l=4$ $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-1+0\\0+4\\-3+8\end{pmatrix} = \begin{pmatrix}-1\\4\\5\end{pmatrix}$ as required