Q1) $\begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = \begin{pmatrix} 50 & -90 \\ 27 & -49 \end{pmatrix} \begin{pmatrix} p_n \\ q_n \end{pmatrix}$

(a) $\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \begin{pmatrix} 50 & -90 \\ 27 & -49 \end{pmatrix} \begin{pmatrix} 60 \\ 33 \end{pmatrix} = \begin{pmatrix} 3000-2970 \\ 1620-1617 \end{pmatrix} = \begin{pmatrix} 30 \\ 3 \end{pmatrix}$ so $q_1 = 3$

$\begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = \begin{pmatrix} 50 & -90 \\ 27 & -49 \end{pmatrix} \begin{pmatrix} 30 \\ 3 \end{pmatrix} = \begin{pmatrix} 1500-270 \\ 810-147 \end{pmatrix} = \begin{pmatrix} 1230 \\ 663 \end{pmatrix}$ $q_2 = 663$

(b) $\det\left(\begin{pmatrix} 50-\lambda & -90 \\ 27 & -49-\lambda \end{pmatrix}\right) = \lambda^2 - 50\lambda + 49\lambda - 2450 + 2430$
$= \lambda^2 - \lambda - 20$
$= (\lambda-5)(\lambda+4)$

$\lambda = 5$ $\underline{v}_1$ $\left(\begin{array}{cc|c} 45 & -90 & 0 \\ 27 & -54 & 0 \end{array}\right)$ $\xrightarrow{R_1 \leftarrow R_1 \times \frac{1}{45},\ R_2 \leftarrow R_2 \times \frac{1}{27}}$ $\left(\begin{array}{cc|c} 1 & -2 & 0 \\ 1 & -2 & 0 \end{array}\right)$ $\xrightarrow{R_2 \leftarrow R_2 - R_1}$ $\left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\underline{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\lambda_2 = -4$ $\underline{v}_2$ $\left(\begin{array}{cc|c} 54 & -90 & 0 \\ 27 & -45 & 0 \end{array}\right)$ $\xrightarrow{R_1 \leftarrow R_1 - 2R_2}$ $\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 27 & -45 & 0 \end{array}\right)$ $\xrightarrow{R_2 \leftarrow R_2 \times \frac{1}{9}}$ $\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 3 & -5 & 0 \end{array}\right)$ $\underline{v}_2 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

so $P = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ $D^k = \begin{pmatrix} 5^k & 0 \\ 0 & (-4)^k \end{pmatrix}$ $P^{-1} = \frac{1}{6-5}\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$

so $\begin{pmatrix} p_k \\ q_k \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 5^k & 0 \\ 0 & (-4)^k \end{pmatrix} \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 60 \\ 33 \end{pmatrix}$

$= \begin{pmatrix} 2\times 5^k & 5\times(-4)^k \\ 5^k & 3\times(-4)^k \end{pmatrix} \begin{pmatrix} 15 \\ 6 \end{pmatrix} = \begin{pmatrix} 30\times 5^k + 30\times(-4)^k \\ 15\times 5^k + 18\times(-4)^k \end{pmatrix}$

(c) $p_k - q_k = 15\times 5^k + 12\times(-4)^k$ since $5^k \geq (-4)^k$ $k \geq 0$
$12\times 5^k \geq -12\times(-4)^k$
so $15\times 5^k > -12\times(-4)^k$

If $\frac{p_0}{q_0} = \frac{5}{3}$ (non dom eigenvector) then $p_0 = \frac{5}{3}\times 33 = 55$

and $p_k = 55\times(-4)^k$ and $-220 < -132$ etc
$q_k = 33\times(-4)^k$

Q2)

$$x^3 - 12x^2 + 29x + 42$$

$x = -1$: $-1 - 12 - 29 + 42 = 0$

$$\begin{array}{r} x^2 - 13x + 42 \\ x+1 \overline{\big)\, x^3 - 12x^2 + 29x + 42} \\ \underline{x^3 + x^2 \qquad\qquad\qquad} \\ -13x^2 + 29x \qquad \\ \underline{-13x^2 - 13x \qquad} \\ 42x + 42 \\ \underline{42x + 42} \\ 0 \end{array}$$

and $x^2 - 13x + 42 = (x-7)(x-6)$

$\lambda_1 = -1 \qquad \underline{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$\lambda_2 = 6 \qquad \underline{v}_2 = \begin{pmatrix} 36 \\ 6 \\ 1 \end{pmatrix}$

$\lambda_3 = 7 \qquad \underline{v}_3 = \begin{pmatrix} 49 \\ 7 \\ 1 \end{pmatrix}$

b)

$$\left(\begin{array}{ccc|c} 1 & 36 & 49 & -539 \\ -1 & 6 & 7 & -105 \\ 1 & 1 & 1 & -3 \end{array}\right)$$

$R_1 \Leftarrow R_1 - R_3 \quad R_2 \Leftarrow R_2 + R_3$

$$\left(\begin{array}{ccc|c} 0 & 35 & 48 & -536 \\ 0 & 7 & 8 & -108 \\ 1 & 1 & 1 & -3 \end{array}\right)$$

$R_1 \Leftarrow R_1 - 5R_2 \quad R_3 \Leftarrow R_3 - \frac{1}{7}R_2$

$$\left(\begin{array}{ccc|c} 0 & 0 & 8 & 4 \\ 0 & 1 & 8/7 & -108/7 \\ 1 & 0 & -1/7 & 87/7 \end{array}\right) \qquad -3 + \frac{108}{7} = \frac{108-21}{7} = \frac{87}{7}$$

$R_2 \Leftarrow R_2 \times \frac{1}{7} \quad R_1 \Leftarrow R_1 \times \frac{1}{8}$

$$\left(\begin{array}{ccc|c} 0 & 0 & 1 & 1/2 \\ 0 & 1 & 8/7 & -108/7 \\ 1 & 0 & -1/7 & 87/7 \end{array}\right)$$

$$-\frac{108}{7} - \frac{4}{7} = \frac{-112}{7} = -16$$

$$\frac{87}{7} + \frac{1}{14} = \frac{174+1}{14} = \frac{25}{2}$$

$R_2 \Leftarrow R_2 - \frac{8}{7}R_1$

$R_3 \Leftarrow R_3 + \frac{1}{7}R_1$

$$\left(\begin{array}{ccc|c} 0 & 0 & 1 & 1/2 \\ 0 & 1 & 0 & -16 \\ 1 & 0 & 0 & 25/2 \end{array}\right)$$

$$M^k \begin{pmatrix} -539 \\ -105 \\ -3 \end{pmatrix} = \begin{pmatrix} \sim\sim\sim \\ \hline (-1)^k \; 6^k \; 7^k \end{pmatrix} \begin{pmatrix} 25/2 \\ -16 \\ 1/2 \end{pmatrix}$$

Thus $a_k = \frac{25}{2} \times (-1)^k - 16 \times 6^k + \frac{1}{2} \times 7^k$

(c) $|\lambda_3| = 7 > |\lambda_1| = 1$ and $> |\lambda_2| = 6$

so as $k \to \infty$

$$a_k \simeq \frac{1}{2} \times 7^k$$

we want $-16 \times 6^k + \frac{1}{2} \times 7^k = 0$

or $\left(\frac{7}{6}\right)^k = 32$

thus $k = \dfrac{\ln(32)}{\ln(7/6)}$

$= \dfrac{\cancel{0.24086}\ 3.466}{0.154}$

$= 22.48$

so $k = 23$