Math 2103 Assignment 4: Cayley-Hamilton and

March 29, 2012

Answer all questions and show all working and check each of your results. Any rough work done before attempting your solutions should be attached to your answers as I need to know how you came up with them. You are allowed to talk with myself or other members of the class in general about the questions, but you must do them on your own.

1. You will be using the matrix M that you selected for this question.

   (a) Use Cholesky factorisation to show that M is not positive definite and hence or otherwise find a vector $\underline{v} := \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ such that $\underline{v}^T M \underline{v} < 0$. [4]

   (b) Calculate $I \times \det(M - xI)$ and substitute $x = M$ into the expanded version to check the Cayley-Hamilton theorem for your M. Plot the polynomial $\det(M - xI)$ to verify that one of the eigenvalues of M is negative. [3]

   (c) For your M create the adjoint of the matrix $(M - xI)$; it should contain various powers of x. Separate the powers so that you have numerical matrices B_j for j from 0 to $n = 3$ such that [4]

$$\mathrm{adjoint}(M - xI) = \sum_{j=0}^{n} B_j x^j$$

   (d) Use the fact that any matrix times its adjoint is its determinant times the identity to get a set of equations for the B_j matrices in general from the equation above by comparing powers of x. Use these equations to prove Cayley-Hamilton for any $n \times n$ matrix A. [6]

2. (a) Calculate the eigenvectors of your Hermitian matrix H, and hence unitarily diagonalise it, verifying that your matrix of eigenvectors is unitary. Get an expression for H^k for any integer k using diagonalisation. [6]

   (b) Modify the proof that the eigenvalues of a symmetric matrix are real and the eigenvectors are orthogonal to prove the same result for Hermitian matrices. [2]

$$M := \begin{bmatrix} 3 & 2 & 4 \\ 2 & 6 & 0 \\ 4 & 0 & 4 \end{bmatrix} \quad , \quad H := \begin{bmatrix} -4 & -3+4\,i \\ -3-4\,i & -4 \end{bmatrix}$$

$$M := \begin{bmatrix} 1 & 4 & 3 \\ 4 & 7 & -1 \\ 3 & -1 & 2 \end{bmatrix} \quad , \quad H := \begin{bmatrix} 2 & 4+2\,i \\ 4-2\,i & 3 \end{bmatrix}$$

$$M := \begin{bmatrix} 6 & -1 & 2 \\ -1 & 1 & 5 \\ 2 & 5 & 7 \end{bmatrix} \quad , \quad H := \begin{bmatrix} -3 & -2+4\,i \\ -2-4\,i & -4 \end{bmatrix}$$

$$M := \begin{bmatrix} 7 & -1 & 7 \\ -1 & 5 & -1 \\ 7 & -1 & 6 \end{bmatrix} \quad , \quad H := \begin{bmatrix} -3 & 2+2\,i \\ 2-2\,i & -1 \end{bmatrix}$$