Math 2103 (2016/17)
Assignment 4: Inner products and orthogonality

Answer all questions and show all working and check each of your results. Any rough work done before attempting your solutions should be attached to your answers as I need to know how you came up with them. You are allowed to talk with myself or other members of the class in general about the questions, but you must do them on your own.

The numbers represented by p, q and r should be replaced by the last three *non-zero* digits of your registration number in that order. For instance, if my registration number was 20015374 then I would take $p = 3$, $q = 7$ and $r = 4$.

1. (a) Given your matrix M use Cholesky to show that it is positive definite and hence express $\begin{pmatrix} x & y & z \end{pmatrix} M \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ as a sum of multiples of squares. [4]

   (b) Use algebra to show that doing the pivot row operations using the first row of any positive definite matrix will give the same coefficients as completing the square of the quadratic form for the first variable and why the resulting matrix (apart from the first row and column) will be symmetric if the original one was. [3]

2. (a) Create a 2×2 Hermitian matrix H with rank one with the first row containing both p and $q + ir$. [1]

   (b) Now defining a complex linear operator $T(\underline{v}) := H\underline{v}$, find the image space of T and the kernel of T. [2]

   (c) Show that they are orthogonal spaces using the complex inner product. [1]

   (d) Explain why, using algebra, that the kernel and image spaces will necessarily be orthogonal for any operator based on an $n \times n$ Hermitian matrix H. [2]

3. Let U be the subspace of $\mathbb{P}_3$ which is spanned by $\{x^3 + px - q, x^2 + rx + q\}$ and we will be using the inner product $< f(x), g(x) >:= \int_0^1 f(x)g(x)\mathrm{d}x$ for this question.

   (a) Normalise the basis vectors of U and create an orthogonal basis for U . [2]

   (b) Determine a basis for $U^\perp$. [3]

   (c) Find a standard basis for $U^\perp$ with integer coefficients explaining why it has the maximum number of zeros possible. [2]

$$M_1 := \begin{pmatrix} 4 & -3 & 2 \\ -3 & 4 & -3 \\ 2 & -3 & 4 \end{pmatrix}$$

$$M_2 := \begin{pmatrix} 4 & -1 & 2 \\ -1 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$$

$$M_3 := \begin{pmatrix} 4 & 2 & -3 \\ 2 & 3 & -2 \\ -3 & -2 & 4 \end{pmatrix}$$

$$M_4 := \begin{pmatrix} 4 & -2 & 1 \\ -2 & 4 & 1 \\ 1 & 1 & 3 \end{pmatrix}$$

$$M_5 := \begin{pmatrix} 4 & -2 & -3 \\ -2 & 3 & 3 \\ -3 & 3 & 4 \end{pmatrix}$$

$$M_6 := \begin{pmatrix} 4 & -1 & -3 \\ -1 & 1 & 1 \\ -3 & 1 & 3 \end{pmatrix}$$