Math 2103 (2016/17)
Assignment 5: Singular Values etc.

Answer all questions and show all working and check each of your results. Any rough work done before attempting your solutions should be attached to your answers as I need to know how you came up with them. You are allowed to talk with myself or other members of the class in general about the questions, but you must do them on your own.

The numbers represented by q and r should be replaced by the last two *non-zero* digits of your registration number in that order.

1. Let A be the 4×2 matrix you chose. We are going to make the Singular Value Decomposition of A and use it to make the Moore-Penrose Inverse.

   (a) Find the eigenvalues and eigenvectors of $A^T A$. [2]

   (b) Find orthogonal homogeneous solutions to $(AA^T)\underline{v} = \underline{0}$ and multiply the eigenvectors from (a) to make all four of the eigenvectors of AA^T. [3]

   (c) Carefully create the eigenvector matrices P and Q such that $P^{-1} = P^T$ and $Q^{-1} = Q^T$ and check that $A = PFQ^T$ as expected for the 4×2 eigenvalue matrix F. [2]

   (d) For your matrix A calculate the matrix $J := QGP^T$ for G such that FG and GF are diagonal matrices with as many 1s as possible along their diagonal. Using algebra show that J has the four properties that A^+ needs to have. [3]

   (e) Use the Rayleigh Quotient method to find how many steps it takes to get the eigenvector of the largest eigenvalue of AA^T to be correct to 2 decimal places. [2]

2. Let M be the $n \times n$ block matrix $\begin{bmatrix} U & \underline{w} \\ \underline{0}^T & \sqrt{b} \end{bmatrix}$. Find an expression for the determinant of M and the block forms of $M^T M$ and MM^T and their determinants. [2]

3. Let V be a finite dimension vector space and S be a linear operator acting on that space. Define S^j to be the operator which is S acting on a vector j times.

   Explain why $W := \text{span}(\{\underline{v}, S(\underline{v}), S^2(\underline{v}), \ldots\})$ is guaranteed to be an S-invariant space for any $\underline{v}$ in V. Find such a space in $\mathbb{P}_3$ which is of dimension 3 by finding and choosing a suitable S which is essentially different from those submitted by other members of the class. Experiments finding subspaces of smaller dimensions will also get you some marks. [4]

4. Find the relations that must hold for a 2×2 matrix to be normal and hence or otherwise find one that has $q + ri$ in the top left corner. [2]

$$A_1 := \begin{pmatrix} -2 & -1 \\ -1 & -2 \\ -2 & 1 \\ -1 & -2 \end{pmatrix} \qquad A_2 := \begin{pmatrix} 1 & -1 \\ 0 & -2 \\ -1 & -1 \\ 2 & -3 \end{pmatrix}$$

$$A_3 := \begin{pmatrix} -2 & 2 \\ 3 & 4 \\ 4 & 1 \\ 4 & -3 \end{pmatrix} \qquad A_4 := \begin{pmatrix} -1 & -2 \\ 2 & -1 \\ -2 & 2 \\ -2 & -2 \end{pmatrix}$$

$$A_5 := \begin{pmatrix} -2 & 3 \\ 1 & 1 \\ 1 & -2 \\ 3 & -1 \end{pmatrix} \qquad A_6 := \begin{pmatrix} -1 & -1 \\ -2 & -2 \\ 2 & 1 \\ 1 & -2 \end{pmatrix}$$