Math 2103 Assignment 4: Inner products

November 6th; Due November 22nd

Answer all questions and show all working and check each of your results. Any rough work done before attempting your solutions should be attached to your answers as I need to know how you came up with them. You are allowed to talk with myself or other members of the class in general about the questions, but you must do them on your own. Maple worksheets can be submitted via email showing the working done. Any submissions which are received d days past the deadline can receive a maximum of $(20-d)$ marks for this assignment.

You have selected a matrix M to deal with in this question.

1. Let $f(x,y,z) := \underline{v}^T M \underline{v}$ where $\underline{v} := \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

   (a) Use either Cholesky or completing the square to find an expression for $f(x,y,z)$ made of a sum of multiples of squares and expand your final expression to show that it is still equal to $f(x,y,z)$. [5]

   (b) Use Maple or another program to get the eigenvalues and eigenvectors of M and verify that the eigenvectors are orthogonal as expected. Take the unit eigenvectors and hence find matrices P and D such that $M = PDP^T$ and verify that $PP^T = I$ by hand. [4]

   (c) Show that the eigenvector of the negative eigenvalue gives values of x, y and z which make $f(x,y,z)$ negative and then experiment to find an integer valued $\underline{v}$ which is not an eigenvector which gives a negative value for $f(x,y,z)$. [2]

2. Let the inner product of two polynomials $g(x)$ and $h(x)$ be defined for this question as $<g(x),h(x)>:= \int_0^1 g(x)h(x)\mathrm{d}x$. You have polynomials $p(x)$, $q(x)$ and $r(x)$ derived from each column $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ of M as ax^2+bx+c. Note that these will give rise to fractional numbers in your answers so I strongly advise you to use a computer to calculate and check them.

   (a) Use Gram-Schmidt on $\{p(x), q(x)\}$ to form an orthogonal set and also find the polynomial $s(x) := x^2+bx+c$ which is orthogonal to both $p(x)$ and $q(x)$ by finding the unique values of b and c such that $<p(x), x^2+bx+c>= 0$ and $<q(x), x^2+bx+c>= 0$. [5]

   (b) Verify that $<q(x),p(x)>\leq ||q(x)|| \times ||p(x)||$ and $||q(x)+r(x)|| \leq ||q(x)|| + ||r(x)||$ for your particular polynomials but then find other non zero polynomials to replace $p(x)$ and $r(x)$ which when combined with $q(x)$ in these formulae have the "$\leq$" actually as "$=$". [4]

$$M_1 := \begin{pmatrix} 5 & -1 & 2 \\ -1 & 1 & 4 \\ 2 & 4 & 5 \end{pmatrix}$$

$$M_2 := \begin{pmatrix} 1 & 1 & 2 \\ 1 & 4 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

$$M_3 := \begin{pmatrix} 5 & 1 & 4 \\ 1 & 1 & 2 \\ 4 & 2 & 1 \end{pmatrix}$$

$$M_4 := \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 2 \\ -1 & 2 & 1 \end{pmatrix}$$