Math 2103 (Fall 2012)
Assignment 4: Inner Products and Orthogonality

Answer all questions and show all working and check each of your results. Any rough work done before attempting your solutions should be attached to your answers as I need to know how you came up with them. You are allowed to talk with myself or other members of the class in general about the questions, but you must do them on your own.

The numbers represented by k, m and n should be replaced by the largest three digits of your registration number. For instance, if my registration number was 20015270 then i would take $k = 7$, $m = 5$ and $n = 2$.

1. (a) Define an inner product on polynomials as follows: $\langle f, g\rangle := \int_0^1 fg\mathrm{d}x$. Use the Gram-Schmidt process to create an orthogonal basis for $\mathbb{P}_2$, starting with this basis set: $\{6x^2 + k, 4x + n, 9x^2 + mx\}$. [3]

   (b) By considering $< \underline{u} + \underline{v}\ ,\ \underline{u} + \underline{v} >$ and using Cauchy-Schwartz and the axioms of real valued Inner Products, show that $||\underline{u} + \underline{v}|| \leq ||\underline{u}|| + ||\underline{v}||$. Why does it still hold even if the complex vector inner product is used? [5]

   (c) Use Cholesky to decide if $A := \begin{pmatrix} n & 4 & -3 \\ 4 & m & -1 \\ -3 & -1 & k \end{pmatrix}$ is a positive definite matrix. Create the quadratic form based on this matrix and either verify or disprove that $\underline{v}^T A\underline{v} > 0$ for $\underline{v} \neq \underline{0}$. [4]

2. (a) Create a 2×2 matrix which has $n + i$ as an eigenvalue and $\underline{w} := \begin{pmatrix} m - 2i \\ -3 \end{pmatrix}$ as an eigenvector. What complex vector is orthogonal to $\underline{w}$? Hence explain why any 2×2 matrix can be written in the form QSQ^* where Q is unitary and S is upper triangular with eigenvalues on the diagonal. [4]

   (b) Use induction on the size of the matrix to show in a similar way that we did to prove orthogonal diagonalisation for symmetric matrices that for any complex matrix C we can write $C = QSQ^*$ as in the previous part. [6]

3. Create a random 3×3 symmetric matrix using your registration number or Maple:
   A :=`RandomMatrix(3,3,generator=rand(-5..5),shape='symmetric');`
   Make sure that the eigenvalues of A are irrational and then find one and its eigenvector correct to 5 decimal places using the iterative method described in class. [3]