Math 226 Test 3: Linear Transformations and Isomorphisms

November 4, 2005

Each question is weighted as shown in square brackets, use the appropriate amount of time and space to answer all parts. Give *all* working and reasoning for each question to achieve full marks.

The answers must be entirely your own work, and a statement to this effect should preface your answers. Plagiarism will be detected and the appropriate academic penalties enforced.

1. (a) Prove that this set of matrices forms a basis for $\mathbb{M}_{2,2}$, the 2×2 matrices.

$$b_1 := \begin{pmatrix} 3 & 0 \\ 3 & 1 \end{pmatrix}, \quad b_2 := \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}, \quad b_3 := \begin{pmatrix} -1 & 1 \\ -2 & 0 \end{pmatrix}, \quad b_4 := \begin{pmatrix} -3 & 1 \\ -3 & 1 \end{pmatrix}$$

   (b) Find the linear transform which maps $\{b_1, b_2, b_3, b_4\}$ to the four vectors of the standard basis in terms of its effect on the general 2×2 matrix.

   (c) By calculating the kernel prove that the linear transform in (b) is an isomorphism.

2. (a) One of these is not a linear transformation; identify it by checking the two linear transformation axioms for all three transformations.

$$R\left(ax^2 + bx + c\right) := \begin{pmatrix} 2a - c & c + b \\ b + 2a & 0 \\ 2c + b - 2a & c - 2a \end{pmatrix}, \quad S\left(\begin{pmatrix} g & h \\ i & j \end{pmatrix}\right) := (g+i-2j)x+(j-g)$$

$$T\left(kx^3 + lx^2 + mx + n\right) := \begin{pmatrix} k + 2l \\ m^2 \div n \\ l + 2n- \end{pmatrix}$$

   (b) Find the kernel and image space of the other two.

3. Prove that if $\{v_1, \ldots, v_n\}$ are independent vectors then

$$\{v_1 - 2v_2 + v_3, 2v_2 - 4v_3 + 2v_4, \ldots, 2^{n-1}v_{n-1} - 2^n v_n + 2^{n-1}v_1, 2^n v_n - 2^{n+1}v_1 + 2^n v_2\}$$

is not but $\{v_1 - v_2 + v_3, v_2 - v_3 + v_4, \ldots v_{n-1} - v_n + v_1, v_n - v_1 + v_2\}$ is.