Math 226 Test 4: Linear Operators and Inner Spaces

November 28, 2005

Each question is weighted as shown in square brackets, use the appropriate amount of time and space to answer all parts. Give *all* working and reasoning for each question to achieve full marks.

The answers must be entirely your own work, and a statement to this effect should preface your answers. Plagiarism will be detected and the appropriate academic penalties enforced.

1. (a) Given this linear operator

$$T(a + bx + cx^2) := (2c - \frac{b}{2}) + (2c - \frac{5b}{2} - a)x + (3c - \frac{3b}{2} - 1)x^2$$

   show that the subspace $U := \text{span}(\{1 - 2x, 3 + 2x^2\})$ is T-invariant.

   (b) Prove that $W := \{k(1 - x)^2; k \in \mathbb{R}\}$ and U have direct sum $\mathbb{P}^2$.

   (c) Prove that W is not T-invariant.

2. (a) Manipluate the rules of inner products to show that

$$\langle v, w \rangle \leq \frac{\langle v, v \rangle + \langle w, w \rangle}{2}$$

   and determine under what circumstances equality occurs.
   [Hint: use the inner product of $u := v - w$ with itself]

   (b) Verify axioms P1 and P4 for this inner product:

$$\left\langle \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \right\rangle := u_1v_1 - 2u_1v_3 + 3u_2v_2 - 4u_2v_3 - 2u_3v_1 - 4u_3v_2 + 10u_3v_3$$

   (c) Using the above inner product with the standard basis to find an orthogonal basis for $\mathbb{R}^3$ using the Gram-Schmidt Method.