Math 226 Assignment 2: Bases and Transformations

Answer all questions and show all working and check each of your results. Any rough work done before attempting your solutions should be attached to your answers as I need to know how you came up with them. You are allowed to talk with myself or other members of the class in general about the questions, but you must do them on your own. The letters a, b, c and d are the last four non-zero digits in your student ID number.

1. (a) For these sets, find which are independent and which span $\mathbb{R}^3$. [6]

$$S_1 := \left\{ \begin{pmatrix} a \\ -4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ b \end{pmatrix} \right\}, \quad S_2 := \left\{ \begin{pmatrix} a \\ -4 \\ 2 \end{pmatrix}, \begin{pmatrix} c \\ 6 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \right\},$$

$$S_3 := \left\{ \begin{pmatrix} a \\ -4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ b \end{pmatrix}, \begin{pmatrix} 2 \\ d \\ -1 \end{pmatrix} \right\}, \quad S_4 := S_1 \cup S_3$$

   (b) Find the vector $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ in terms of a linear combination of one of your sets, and explain why this isn't possible for another. [2]

   (c) Create a linear transformation T from $\mathbb{P}_2$ to $\mathbb{R}^3$ which has S_1 as the image space of T and whose kernel contains $x^2 + dx + c$. [3]

2. Let $\mathbb{A}_n$ be the set of $n \times n$ skew-symmetric matrices, that is real valued matrices A which satisfy $A^T = -A$. Let $\mathbb{B}_n$ be the set of $n \times n$ symmetric matrices, that is real valued matrices B which satisfy $B^T = B$.

   (a) Verify that $\mathbb{A}_n$ and $\mathbb{B}_n$ are subspaces of the space of $n \times n$ matrices $\mathbb{M}_n$. [2]

   (b) Find standard bases for $\mathbb{A}_n$ and $\mathbb{B}_n$ for $n = 2$ and 3 and generalise to find their dimensions for an arbitrary n. [3]

   (c) Give a simple linear transformation from $\mathbb{B}_n$ onto $\mathbb{A}_n$ and determine its kernel after checking both linear transformation axioms. [3]

3. Prove that, in a vector space V, for any two subspaces U and W the intersection $U \cap W$ and the sum $U + W := \{\underline{u} + \underline{w} \mid \underline{u} \in U, \underline{w} \in W\}$ are also subspaces of V. Explain how the dimensions of these two spaces will be related together with the dimensions of U and W. [6]