Math 226 2011 Assignment 2: Vector Spaces and Transformations

Answer all questions and show all working and check each of your results. Any rough work done before attempting your solutions should be attached to your answers as I need to know how you came up with them. You are allowed to talk with myself or other members of the class in general about the questions, but you must do them on your own.

The numbers represented by a, b and c should be replaced by the last three digits of your registration number in that order, and if the digit is 0 it is replaced by 10. For instance, if my registration number was 20015270 then i would take $a = 2$, $b = 7$ and $c = 10$.

1. Define a linear transformation T as follows:

$$T(\begin{pmatrix} p & q \\ r & s \end{pmatrix}) = \begin{pmatrix} p - aq \\ br - s \\ cq + cr \\ bp + as \end{pmatrix}$$

   Feel free to work in either $\mathbb{Z}_{13}$ or in $\mathbb{R}$, whichever you prefer.

   (a) Determine $\ker(T)$ and hence or otherwise find two matrices that map to $\begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$. [5]

   (b) Get a basis for the image space of T and check that the dimension theorem holds. [3]

2. (a) Prove using any combination of vector space axioms (apart from commutativity A2) that A4 and A5 both hold in the opposite orders, i.e. $(-\underline{v}) + \underline{v} = \underline{0}$ and $\underline{0} + \underline{v} = \underline{v}$. [5]

   (b) These can be used to prove cancellation works on either side of a vector equation (that $\underline{w} + \underline{v} = \underline{w} + \underline{u}$ implies that $\underline{v} = \underline{u}$ and $\underline{v} + \underline{w} = \underline{u} + \underline{w}$ implies that $\underline{v} = \underline{u}$). Use cancellation and part (a) to prove A2 by considering the expansion of $(1 + 1)(\underline{u} + \underline{w})$ in two ways. [4]

3. Working in $\mathbb{P}_2$ let $\underline{v}_1 := e_1x^2 + f_1x + g_1$ and $\underline{v}_2 := e_2x^2 + f_2x + g_2$. Define vector addition and scalar multiplication as follows:

$$\begin{aligned} \underline{v}_1 + \underline{v}_2 &:= (e_1 - e_2)x^2 + (f_1 + g_2)x + (g_1 - \frac{1}{2}f_2) \\ \alpha\underline{v}_1 &:= \alpha e_1 x \end{aligned}$$

   (a) Prove that axiom A4 holds and the zero vector is $0x^2 + 0x + 0$. Show that A5 also holds and find $(-\underline{v}_1)$ under these definitions. [4]

   (b) For which space of vectors are axioms S2 and S3 true? [4]