Math 226 2011 Assignment 3: Isomorphisms and Linear Operators

Answer all questions and show all working and check each of your results. Any rough work done before attempting your solutions should be attached to your answers as I need to know how you came up with them. You are allowed to talk with myself or other members of the class in general about the questions, but you must do them on your own.

The numbers represented by a, b and c should be replaced by the last three digits of your registration number in that order, and if the digit is 0 it is replaced by 10. For instance, if my registration number was 20015270 then i would take $a = 2$, $b = 7$ and $c = 10$. Feel free to work in either $\mathbb{Z}_{13}$ or in $\mathbb{R}$, whichever you prefer.

1. (a) Create two different isomorphisms from the space of 2×2 matrices to the cubic polynomials which both map these basis vectors as shown. [4]

$$\begin{pmatrix} 1 & 3 \\ 0 & a \end{pmatrix} \to bx^3 - x + 1 \ , \quad \begin{pmatrix} 2 & c \\ 0 & 5 \end{pmatrix} \to (x-2)(x+3)$$

   (b) Show they are isomorphisms by calculating the kernels and show they are different by examining their effect on a particular vector. Explain exactly which vectors in the space they will be different for. [5]

   (c) Find the standard form of the inverse of one of your isomorphisms and get a linear operator from the composition of it with the other isomorphism. [4]

2. Given two vector spaces V and W, what dimensions can they be if the composition of two linear transformations between them is an isomorphism? Give an example where $\dim(V) \neq \dim(W)$. [3]

3. Choose any two vectors $\underline{u}$ and $\underline{w}$ as a non-standard basis for a space V different from everyone else in the class and define $T(\underline{u}) = -\underline{w}$ and $T(\underline{w}) = 2\underline{u}$. Get the transformation of any vector in V and find all T-invariant subspaces of dimension 1. Can you find a 3-dimensional space and a linear operator S such that there are no 1 or 2 dimensional S-invariant subspaces? [6]

4. Explain why matrix transpose is a linear operator on $\mathbb{M}_{n,n}$ and determine the largest invariant subspace under it. Is determinant a linear transformation from the same space? [3]