UNIVERSITY OF ZIMBABWE

Bsc Honours Part II, General Part III Mathematics

Graph Theory

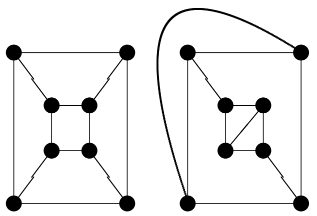
November 1994
Time : 2 hours

Candidates may attempt ALL questions in Section A and at most TWO questions in Section B. Each question should start on a fresh page.

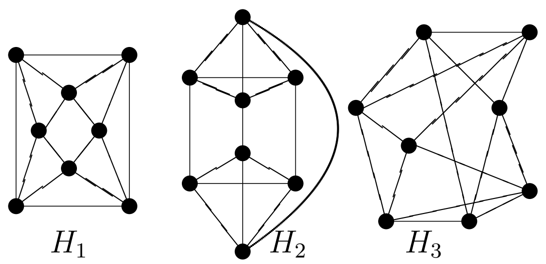
SECTION A (40 marks)

Candidates may attempt ALL questions being careful to number them A1 to A5.

A1. The *girth* of a graph G is the length of the shortest circuit in G: what are the girths of the five graphs below ? Giving reasons, prove whether G_1 and G_2 are isomorphic or not. Similarly, show which pairs of H_1, H_2 and H_3 are isomorphic. [9]



G_1 G_2

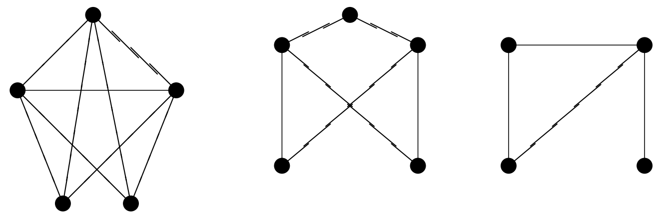


H_1 H_2 H_3

A2. Prove that, in a tree T, there must be at least one vertex of valency one. Use this to prove that $|E(T)| = |V(T)| - 1$. A *caterpillar* is a tree in which every vertex either lies in a longest path P or is joined to P by a single edge. The length of a caterpillar is the length of P. Find a necessary and sufficient condition for a graph G to be a caterpillar. Prove that a caterpillar has a 2-ordering (i.e. the square of a caterpillar is Hamiltonian). [10]

A3. Prove that in a simple graph there have to be at least two vertices of the same valency. Find all graphs on six vertices which have exactly two vertices with the same valency. What is a regular graph ? Find all regular graphs on six vertices. [8]

A4. (a) What are the chromatic polynomials of the graphs K_n and T_n (the complete graph and a tree on n vertices) ? Use these and deletion-contraction and addition-identification to find the chromatic polynomials of the graphs G_1, G_2 and G_3 shown below. [6]



G_1 G_2 G_3

(b) Using induction, find the chromatic polynomial of C_n, the circuit graph on n vertices. [2]

A5. What is a tournament in graph theory ? Prove that there always exists a (directed) Hamiltonian path in any tournament. [5]

SECTION B (60 marks)

Candidates may attempt TWO questions being careful to number them B6 to B9.

B6. (a) Give definitions for these graph theoretical terms:
bipartite, complementary, disconnected, self-complementary.
Show that a simple graph G and its complement cannot both be disconnected. For which values of $|V(G)|$ it is not possible for G to be self-complementary ? (prove your answer) [9]

(b) How many simple graphs are there on one, two, three and four vertices respectively ? Which of these are *not* bipartite ? Which are self-complementary ? Completely characterize all graphs which are both self-complementary *and* bipartite. [11]

B7. (a) Prove Dirac's result that if $\rho(v) \geq d \geq 2$ for all vertices v of a simple graph G then G contains a circuit of length at least $d + 1$. [5]

(b) Given that G is a simple graph on $n \geq 3$ vertices prove both of these results: [12]

(i) if $\rho(v) \geq \frac{1}{2}n \;\forall\; v \in G$ then G is Hamiltonian.

(ii) if $\rho(u) + \rho(v) \geq n$ for all vertices u and v such that u and v are not adjacent in G then G is Hamiltonian.

(c) Exhibit three Hamiltonian graphs: one which doesn't satisfy either (b)(i) or (b)(ii), one which just satisfies (b)(ii), and one which satisfies both. [3]

B8. (a) Prove, using induction, that if G is a planar graph with n vertices, m edges, f faces and c components then $n - m + f = c + 1$. Thus deduce the number of faces of the plane graph G with vertex and edge sets $\{s, t, u, v, w, x, y, z\}$ and $\{uv, ux, uz, tv, vw, xw, yx, yw, ys, st, sz, zt\}$. Find a planar embedding of this graph and verify your answer. If ϕ_i is the number of faces bounded by i edges find all values of ϕ_i for G. Prove, in general, that if G is a connected planar cubic graph (cubic means $\rho(v) = 3$ for all $v \in G$) then

$$3\phi_3 + 2\phi_4 + \phi_5 - \phi_7 - 2\phi_8 - 3\phi_9 - \ldots = 12.$$

[12]

(b) What is the Euler-Poincaré characteristic equation ? Embed K_6 on the projective plane and K_7 on the torus, and hence (or otherwise) give the Euler-Poincaré characteristic of the projective plane and the torus. [8]

B9. (a) The *odd graph* O_k is defined as follows:
Let S be a set of cardinality $2k-1$. The vertices of O_k correspond to the subsets of S of cardinality $k-1$ and two vertices are adjacent if and only if the corresponding subsets are disjoint.
Find O_2 and O_3. Prove that O_k is k-regular.
The *line graph* of a graph G is the graph $L(G)$ with vertex set $E(G)$ such that two vertices of $L(G)$ are adjacent if and only if the corresponding edges in G meet at a vertex. What is the complement of the line graph of K_n for $2 \leq n \leq 5$? [12]

(b) What are the adjacency matrices and characteristic polynomials of K_n for $n = 1, 2$ and 3 ? Calculate the spectra of these graphs. By consideration of the eigenvectors or otherwise, find the spectrum of K_n. [8]

END OF QUESTION PAPER