Math315 Assignment 2

February 13, 2007

Answer all questions and give complete reasons and checks for your answers. Each question or roman numbered part of the first one is worth around the same marks. Start a fresh side of paper for each question. Hand in your rough working together with your final answers. You are reminded that plagiarism is a serious offense and if caught you will suffer the penalties specified by the University.

1. The (unsigned) Stirling numbers of the first kind are denoted by $\left[\begin{array}{c} n \\ k \end{array} \right]$ and are spoken "n cycle k". They count the number of permutations of the symbols $\{1, \ldots, n\}$ which include exactly k cycles.

   (a) Explain why the following relations hold:

   $$\left[\begin{array}{c} n \\ n \end{array} \right] := 1 \, , \quad \left[\begin{array}{c} n \\ n-1 \end{array} \right] := \left(\begin{array}{c} n \\ 2 \end{array} \right) \, , \quad \left[\begin{array}{c} n \\ 1 \end{array} \right] := (n-1)!$$

   and prove a formula for $\left[\begin{array}{c} n \\ n-2 \end{array} \right]$.

   (b) Build up a triangle containing all non-zeros entries by listing the permutations with $n \leq 4$ and explain why the zeros surround the triangle.

   (c) Explain why the numbers satisfy this recurrence relation and complete the next two rows of the triangle using it.

   $$\left[\begin{array}{c} n \\ k \end{array} \right] = \left[\begin{array}{c} n-1 \\ k-1 \end{array} \right] + n \left[\begin{array}{c} n-1 \\ k \end{array} \right]$$

   (d) Find the factorised generating function for

   $$C_n(x) := \sum_{i \geq 0} \left[\begin{array}{c} n \\ i \end{array} \right] x^i$$

2. Find the differential equtaion of a generating function which could be used to solve this recurrence.

   $$2a_{n-1} - a_n = na_n$$

3. Prove, using induction on k or otherwise, that

   $$\sum_{j \geq 0} \left(\begin{array}{c} j \\ k \end{array} \right) x^j = \frac{x^k}{(1-x)^{k+1}}$$

4. Use the double summation technique to find the generating function for

   $$f(n) := \sum_k \binom{n-ak}{bk} c^k$$

   where a, b and c are the last non-zero digits of your registration number.