Math 3205 Assignment 1, September 2011

Clearly write your answers to the questions showing all reasons, working and checks and indicate what each mathematical calculation is doing. Do not erase anything. Include all rough work and do not commit plagiarism. Feel free to write explanations of what you are thinking at each stage, nothing you can write can lose you marks!

1. (a) Logically list all of the 28 ways that you can choose a set of 6 letters from the choices of the first three letters of your first name. [2]

   (b) For each of these give an interpretation of the set in terms of at which points the letters change, so that "JJAAAM", for instance, could also be represented as "PPNPPPNP" where "P" means print the current letter and "N" means move to the next letter. [2]

   (c) Using this idea, explain in general why the number of ways to choose a set of r letters from an alphabet of size n when you are allowed to repeat letters is: [4]

$$\binom{r+n-1}{r}$$

2. It is true that the only Catalan numbers which are odd is when it is C_{2^k-1}.

   (a) Check this statement for $k = 0, 1, 2, 3$ and determine C_{11} using the formula involving the binomial coefficient. [2]

   (b) Write out the terms of C_{15} in terms of the given sum of products of the previous Catalan numbers and explain why it must be odd, and how you can extend this idea for all C_{2^k-1}. [4]

   (c) Use the same idea to explain why C_{2j} will always be even and check that this same result also comes from the formula linking C_{n+1} and C_n. [2]

3. We are going to investigate this statement:

$$\binom{n}{r} \times \binom{r}{k} = \binom{n}{k} \times \binom{n-k}{n-r}$$

   (a) Suppose your registration number's sixth digit (fifth for Evan) is a. Verify our relation is true for $n = 7$, $r = a$ and $k = 2$. Give two different ways to choose r different objects from a set of n and then all the ways to choose 2 from those r. Use a similar idea to link each of these pairs of sets with a pair of choices which will correspond to the expression on the right side. [3]

   (b) Thinking combinatorially now, and using the ideas from part (a) try to explain in words and with examples why these two expressions are the same quantity. [3]

   (c) Multiply out the factorial formulae on both sides and check they give the same for any values which satisfy $n > r > k$. What happens if $n = r$ or $r = k$? How does this fit with your combinatorial argument? [3]