Math 3207 Assignment 3a, late October 2012

Clearly write your answers to the questions showing all reasons, working and checks and indicate what each mathematical calculation is doing. Do not erase anything. Include all rough work and do not commit plagiarism. Feel free to write explanations of what you are thinking at each stage, nothing you can write can lose you marks!

1. (a) Explain why, for the Möbius function and $n > 1$ we have this formula:

$$\sum_{d|n} \mu(d) = 0.$$

   Check it for three consecutive non-prime numbers that nobody else picks. [3]

   (b) Prove that this equation holds for any multiplicative function f by using induction on the number of primes p_i that divide n. What does it simplify to give when $f = \tau$? [3]

$$\sum_{d|n} \mu(d)f(d) = \prod_{p_i|n} (1 - f(p_i)).$$

   (c) Prove that for any non-trivial multiplicative function $g(n)$ that $g(1) = 1$. If $g(2) = 2$ and for any $j > k$ we have $g(j) > g(k)$, set $g(3) = 3 + t$, derive an inequality for $g(5)$ and hence show that $g(n) = n$. [3]

2. We want to find all numbers m which satisfy this equation:

$$\frac{\phi(m)}{m} = \frac{4}{7}$$

   Suppose that q is the largest prime that divides into m. Use divisibility to get an upper bound for q and thus explain what sorts of numbers m can be. [4]

3. (a) Find a primitive root mod 23 which is not 21 and that nobody else in the class has selected and make a table of powers of it. [2]

   (b) Use your primitive root to make a table of indices and hence solve these equations. If your table of indices doesn't work, you can use the one below which is for 21. [7]

$$z^7 \equiv 19 \pmod{23}\ , \quad 13^y \equiv 4 \pmod{23}\ , \quad x^2 + 17x + 3 \equiv 0 \pmod{529}$$

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
ind j	0	12	8	2	17	20	15	14	16	7	21	10	18	5	3	4	9	6	13	19	1	11

   (c) If p is an odd prime and a is a primitive root mod p^α, determine when a is a primitive root mod $2p^\alpha$ and give a different primitive root when it isn't, hence showing that the number of primitive roots is the same in both moduli. [3]

Math 3207 Assignment 3b, late October 2012

Clearly write your answers to the questions showing all reasons, working and checks and indicate what each mathematical calculation is doing. Do not erase anything. Include all rough work and do not commit plagiarism. Feel free to write explanations of what you are thinking at each stage, nothing you can write can lose you marks!

1. (a) Explain why, for the Möbius function and $n > 1$ we have this formula:

$$\sum_{d|n} \mu(d) = 0.$$

Check it for three consecutive non-prime numbers that nobody else picks. [3]

(b) Prove that this equation holds for any multiplicative function f by using induction on the number of primes p_i that divide n. What does it simplify to give when $f = \sigma$? [3]

$$\sum_{d|n} \mu(d) f(d) = \prod_{p_i|n} (1 - f(p_i)).$$

(c) Prove that for any non-trivial multiplicative function $g(n)$ that $g(1) = 1$. If $g(2) = 2$ and for any $j > k$ we have $g(j) > g(k)$, set $g(3) = 3 + t$, derive an inequality for $g(5)$ and hence show that $g(n) = n$. [3]

2. We want to find all numbers m which satisfy this equation:

$$\frac{\phi(m)}{m} = \frac{3}{7}$$

Suppose that q is the largest prime that divides into m. Use divisibility to get an upper bound for q and thus explain what sorts of numbers m can be. [4]

3. (a) Find a primitive root mod 23 which is not 21 and that nobody else in the class has selected and make a table of powers of it. [2]

(b) Use your primitive root to make a table of indices and hence solve these equations. If your table of indices doesn't work, you can use the one below which is for 21. [7]

$$z^7 \equiv 19 \pmod{23}, \quad 13^y \equiv 4 \pmod{23}, \quad x^2 + 17x + 3 \equiv 0 \pmod{529}$$

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
ind j	0	12	8	2	17	20	15	14	16	7	21	10	18	5	3	4	9	6	13	19	1	11

(c) If p is an odd prime and a is a primitive root mod p^α, determine when a is a primitive root mod $2p^\alpha$ and give a different primitive root when it isn't, hence showing that the number of primitive roots is the same in both moduli. [3]