Math4101 Graph Theory: Assignment 2 (February 2015)

Please show all working and reasoning to get full marks for any question. Hand in your rough working as well so I can see how you investigated and reached your final results. You are reminded that plagiarism is a serious offense and when it is detected you will be punished.

1. (a) Choose a connected bipartite graph (which is not $K_{a,b}$) with at least 4 vertices, calculate its eigenvalues and eigenvectors and verify that the sum of the entries of the neighbours of each vertex of the eigenvector is the correct multiple. Prove in general that if any bipartite graph has eigenvalue λ with eigenvector v then it also has an eigenvalue $-\lambda$ by modifying v in an appropriate way. [4]

   (b) Explain why $K_{a,b}$ has only 3 different eigenvalues (assuming $\max(a, b) > 1$) by finding all of its eigenvectors, utilising the symmetry of the graph. [2]

2. (a) Try to build a 5-regular graph of diameter 2, describing all choices you make and indicate the points at which a triangle or square is forced to be formed. [3]

   (b) Prove that the maximum number of vertices in an r-regular graph of diameter d is $\frac{r(r-1)^d-2}{r-2}$ and find a similar upper bound for r-regular *bipartite* graphs. Determine the largest 3-regular bipartite graph with diameter 3 and find all of the different lengths possible for cycles in it. [5]

   (c) Count the number of different types of 5-cycles in the Hoffman-Singleton graph with respect to how they pass through the base pentagrams and pentagons and hence determine how many 5-cycles there are in the Hoffman-Singleton graph. [2]

3. (a) Determine the labelled tree which corresponds to your registration number using the labels $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Remove the labels and determine the eccentricity of each vertex and hence the radius, diameter and centre. [3]

   (b) Prove that in any graph G that if $uw \in E(G)$ then $|\varepsilon(u) - \varepsilon(w)| \leq 1$. Give an example of a graph (unique within the class) which includes a vertex with neighbours of all possible eccentricities. [3]

   (c) Explore whether or not any graph can be the centre of another graph, giving examples, proofs and intuitions. [3]