Math 4103 (2017/18)
Assignment 4: Conjugation and Cycle Structure

Answer all questions and show all working and check each of your results. Any rough work done before attempting your solutions should be attached to your answers as I need to know how you came up with them. You are allowed to talk with myself or other members of the class in general about the questions, but you must do them on your own. You will randomly pick a slip of paper with the key numbers and facts for your individualised assignment.

1. (a) Using your transposition t, multiply each element in A_4 by it one one particular side and check that all odd permutations in S_4 are created. Use this idea to prove that $|S_n| = 2 \times |A_n|$ by showing multiplication by an odd permutation gives a 1-1 and onto function from A_n. [4]

   (b) Multiply t and r to recursively create 11 different 3-cycles in A_5, give them in canonical form and as a product of combinations of t and r. [4]

   (c) Conjugate your 5-cycle r by your transposition t to produce another 5-cycle q. Verify that r and q belong to different conjugacy classes in A_5 by showing that the implicit conjugations between r and any of the 5 different cycle orderings of q all give rise to odd permutations. [4]

2. We will now look at the general question of conjugacy classes in A_n and S_n for any $n \geq 4$.

   (a) Suppose we have an odd permutation τ which commutes with an even permutation ϕ.

      i. Prove that this commutativity implies that all elements of the same cycle structure as ϕ are conjugate in A_n by showing that any conjugate of ϕ by any odd permutation can also be expressed as a conjugate by an element of A_n. [2]

      ii. Prove the converse, that if $\omega\phi\omega^{-1} = \gamma\phi\gamma^{-1}$ where ω is odd and γ is even then there exists an odd permutation which commutes with ϕ. [1]

   (b) Explain why if $\phi \in A_n$ contains either an even cycle or a pair of cycles of the same length (even if that length is 1) then we necessarily have an odd permutation which commutes with ϕ. [2]

   (c) Finally, if ϕ is a permutation containing any number of odd cycles, no two of the same length, give some examples to help explain how any α such that $\alpha\phi\alpha^{-1} = \phi$ must map every cycle in ϕ to itself. Deduce from this that α is also a permutation containing powers of the same odd cycles as ϕ and hence α must be in A_n. [3]

Illya: t:=(2 3), r:= (1 5 2 4 3)

Courtney: t:=(2 4), r:= (1 2 5 3 4)

Sarah: t:=(3 4), r:= (1 4 2 3 5)