Math415 Graph Theory: Assignment 3 (November 2007)

Please show all working and reasoning to get full marks for any question. Hand in your rough working as well so I can see how you investigated and reached your final results. You are reminded that plagiarism is a serious offense and if it is detected you will be punished.

1. (a) Prove a formula relating every eigenvalue λ of the adjacency matrix of an r-regular graph with an eigenvalue of its Laplacian matrix. [3]

   (b) For this graph, use Maple to find the eigenvalues and verify the formula given in class for the number of labelled spanning trees. Calculate the characteristic polynomial and check that the coefficients identify the number of edges and triangles as claimed. [4]

2. (a) Using the algebraic formula that guarantees non-planarity, investigate for which graphs G can $G + K_2$ possibly be planar. [3]

   (b) By considering minors in such graphs, completely characterise the only planar graphs of the form $G + K_2$. [6]

   (c) Reusing the above working, what graphs of the form $G + \overline{K_2}$ are planar? [2]

3. (a) Prove that a graph with maximum valency r can be properly coloured using $r+1$ colours. Which graphs need exactly this amount? [4]

   (b) What is the smallest chromatic number possible for an r-regular graph? [2]

   (c) What is the smallest chromatic number for a finite, planar r-regular graph? [3]

4. Choose a simple graph H with 9 vertices and 24 edges embedded on the projective plane.

   (a) Re-embed H in the torus and label and count the faces and face sizes in both embeddings. Check the values of $n - m + f$ match the expected ones in both cases. [4]

   (b) Prove that H is non-planar by identifying a Kuratowski minor in it. [2]