Graph Theory
(Math415)
Problems Booklet

dr. j. preen
Q1. Draw the following graph:

\[ V(G) := \{ t, u, v, w, x, y, z \} \text{ and } E(G) := \{ uv, tz, vz, xz, yw, ux \}. \]

Which of these vertices are terminal or isolated?

Q2. Suppose that \( G \) is a 3-regular graph with 10 vertices. How many edges does it have? Does such a \( G \) exist with 11 vertices? Prove that if \( G \) is \( r \)-regular and \( r \) is odd then \( G \) has an even number of vertices.

Q3. Which of these graphs are isomorphic and which are not? (give all your reasoning)

![Graphs]

Q4. Test the following sequences using the theorem from the notes to see if they are graphical or not: (if they are graphical draw an exemplar graph having that valency sequence)

(a) \( (5, 5, 3, 3, 2, 2, 2) \)
(b) \( (8, 6, 5, 4, 3, 2, 2, 2) \)
(c) \( (4, 3, 3, 3, 2, 2, 2, 1) \)
(d) \( (5, 4, 3, 2, 1, 1, 1, 1, 1, 1, 1) \)

Q5. Prove whether or not the following pairs of graphs in the figure above are isomorphic:
Q6. Which are the eleven graphs on four vertices? Which of these are not bipartite?

Q7. Find all graphs with valency sequence (4, 2, 2, 1, 1).

Q8. What is the valency sequence of the graph in question 1? How many components does it have? Are there any different simple graphs with the same valency sequence? Draw those that you find.

Q9. How many vertices and edges are there in $K_{i,j}$ and $K_{i,j,k}$?

Q10. Let $G := K_1 \cup K_2$ and $H := C_4$. Find $G + H$, $H \circ G$, $G \circ H$ and $G \times H$.

Q11. Let $G$ and $H$ have valency sequences (3, 3, 2, 1, 1) and (4, 2, 2, 1, 1). Can you say what the valency sequences of $G + H$, $H \circ G$, $G \circ G$ and $G \times H$ are?

Q12. Which of the graphs in this figure are bipartite (prove your answers)?
Q13. If \( G_i \) has \( n_i \) vertices and \( m_i \) edges for \( i = 1, \ldots, 3 \), how many edges and vertices do these graphs have?

\[
(G_1 \cup G_2) \times G_3 \quad G_1 \circ (G_2 \times G_3) \quad (G_1 + G_2) + G_3 \quad (G_1 \times G_2) + G_3
\]

Q14. What are the complements of these graphs?

\( P_5, \; C_6, \; K_3 \times K_2, \; K_{4,2} \)

Q15. Using the lemmas in the notes identify the self-complementary graphs on less than four vertices.

Q16. Find a self-complementary graph you haven’t already seen.

Q17. Show that a self-complementary graph cannot have a vertex connected to either all or none of the other vertices.

Q18. Prove that regular graphs are reconstructable. [hint: show that the deck of a regular graph is recognizable, and then explain how to reconstruct it]

Q19. Which graph has the deck in the figure below?

Q20. Are there graphs with the following valency sequences? :

\[
(2n+1, 2n+1, 2n-1, 2n-1, \ldots, 3, 3, 1, 1),
(2n, 2n-2, \ldots, 4, 3, 2, 2, 1)
(2n, 2n-1, \ldots, n+1, n, n, n-1, \ldots, 3, 3, 1, 1)\
\]

Q21. Prove that if there exists a walk between two vertices then there is a path between them too.

Q22. Show that if \( G \) has \( n \) vertices and \( \min\{\rho(v); v \in V(G)\} \geq (n - 1)/2 \) then \( G \) is connected.
Q23. Find the eccentricities of all the vertices and hence mark the centres and peripheries of the graphs in the figure above.

Q24. If the square of a graph $G$, $G^2$, is defined as the supergraph of $G$ which has an edge between $u$ and $v$ if and only if $u$ and $v$ are a distance at most two apart in $G$, find $C_5^2$, $P_6^2$, $K_n^2$ and $K_{m,n}^2$.

Q25. Show that in a connected graph the radius and diameter of a graph $G$ satisfy $r(G) \leq d(G) \leq 2r(G)$. Give a graph for which $r(G) = d(G)$ and another which satisfies $d(G) = 2r(G)$.

Q26. Is it possible that a graph can have vertices of eccentricity $e - 1$ and $e + 1$ but no vertex of eccentricity $e$? [either prove or give a counter-example]

Q27. Find a graph which doesn’t have a centre isomorphic to either $K_1$ or $K_2$. Is there any limit upon the size of the centre of a graph? Does the centre of a graph have to be connected?

Q28. Use Kruskal’s algorithm to find a least weight path on the graph in this figure.

Q29. What is the connectivity of $K_{m,n}$, $C_n$, $W_n$, Petersen, a path, a tree?

Q30. [G.A. Dirac, 1952] A finite, simple graph $G$ which has $\rho(v) \geq d \geq 2 \forall v \in G$ has a circuit of length at least $d + 1$.

Q31. If $T$ is a finite tree then $|V(G)| = |E(G)| + 1$. (Hint: use induction and a theorem from the notes)
Q32. Identify which graph representing matrix is used in each of these questions and thus draw the graphs represented by these matrices:

\[
\begin{pmatrix}
0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
\end{pmatrix}, \quad 
\begin{pmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 0 & 1 \\
1 & 0 & 1 \\
\end{pmatrix}, \quad 
\begin{pmatrix}
0 & 1 & 1 & 0 & 0 \\
1 & 0 & 2 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}, \quad 
\begin{pmatrix}
1 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

Q33. We define the line graph, \(L(G)\), of a graph \(G\) as the graph formed from \(G\) by replacing each edge of \(G\) by a vertex and joining two vertices if the two edges they represent were adjacent in \(G\). Find \(L(C_5)\), \(L(P_5)\) and \(L(K_n)\).

Q34. Prove by a constructive argument that \(K_5\) and \(K_{3,3}\) are non-planar.

Q35. Show that these graphs are non-planar by finding a subgraph homeomorphic to \(K_{3,3}\) in them: Petersen, \(P_4 + P_3\), \(K_{2,3,2}\).

Q36. Embed \(K_6\) and \(P_3 \times P_3\) on the projective plane. Clearly mark the faces of each graph and hence count their number.

Q37. Embed \(K_7\), \(K_{4,4}\) and \(P_4 \times P_3\) on the torus. Find the relationship between numbers of faces, edges and vertices.

Q38. Use the algorithm in the notes to construct a plane embedding of this graph:

\[
V(G) := \{a, b, c, d, e, f, g\}
\]

\[
E(G) := \{ab, ad, ae, af, bc, bd, be, bg, cd, ce, cf, cg, df, ef, eg\}
\]

Then use the theorem of Wagner to redraw it so that every edge is a straight line but it is still plane.
Q39. Given that the length of the smallest circuit in a planar graph $G$ (its girth) is $g$, prove that $m \leq \frac{n}{g-2}(n-2)$. [hint: use the same ideas as the result $m \leq 3n - 6 = 3(n - 2)$]

Q40. Draw the five regular polyhedra. Create a new graph from these (their dual graph) by replacing each face in each of them by a vertex and joining those vertices whose faces were adjacent in the original graph. To which graphs are the duals isomorphic?

Q41. What is the (vertex) chromatic number of $K_n$, $K_{m,n}$, $C_n$, $W_n$, a tree and the Petersen graph? (make sure that the number of colours used is the minimum necessary)
What is the edge chromatic number of these graphs? (their chromatic index)

Q42. Calculate the chromatic polynomials of the graphs in the figure at the top of the page.

Q43. What are $\chi(C_n)$, $\chi(W_n)$ and $\chi(T)$? (T a tree)
[hint: start from small graphs and try to find a general formula]

Q44. Show that for an $n$ vertex graph $G$ these equations hold:

\[
2\sqrt{n} \leq \chi(G) + \chi(G) \leq n + 1 \\
\chi(G)\chi(G) \geq n
\]

Q45. If $G$ has $n$ vertices and the size of the largest complete subgraph of $G$ is $q$ (G's clique size) then show that $\chi(G) \geq \frac{n}{q}$.

Q46. If $G$ has $n$ vertices and the smallest valency in $G$ is $\delta$ show that $\chi(G) \geq \frac{n}{n-\delta}$.
Q47. Prove that Fleury’s algorithm does indeed give an Euler-circuit. Use the algorithm to find such a circuit in the graph in the figure above.

Q48. In what circumstances can an arbitrary collection of dominoes be arranged in a line as shown (14100...)? For which \(n\) can the whole set of dominoes labeled with numbers 0 to \(n\) be so arranged? [dominoes are wooden rectangles with two numbers on each end]

Q49. Describe (or draw) a graph on six vertices which is Eulerian and Hamiltonian and describe an Euler circuit and a Hamiltonian circuit in it. Do the same for Eulerian but not Hamiltonian, not Eulerian but Hamiltonian and neither Eulerian nor Hamiltonian.

Q50. Find a graph which isn’t Hamiltonian but which has a solution to the Grinberg’s theorem equation.

Q51. Which of these graphs are Hamiltonian? Mark a Hamiltonian path (if one exists) if the graph is not hamiltonian.