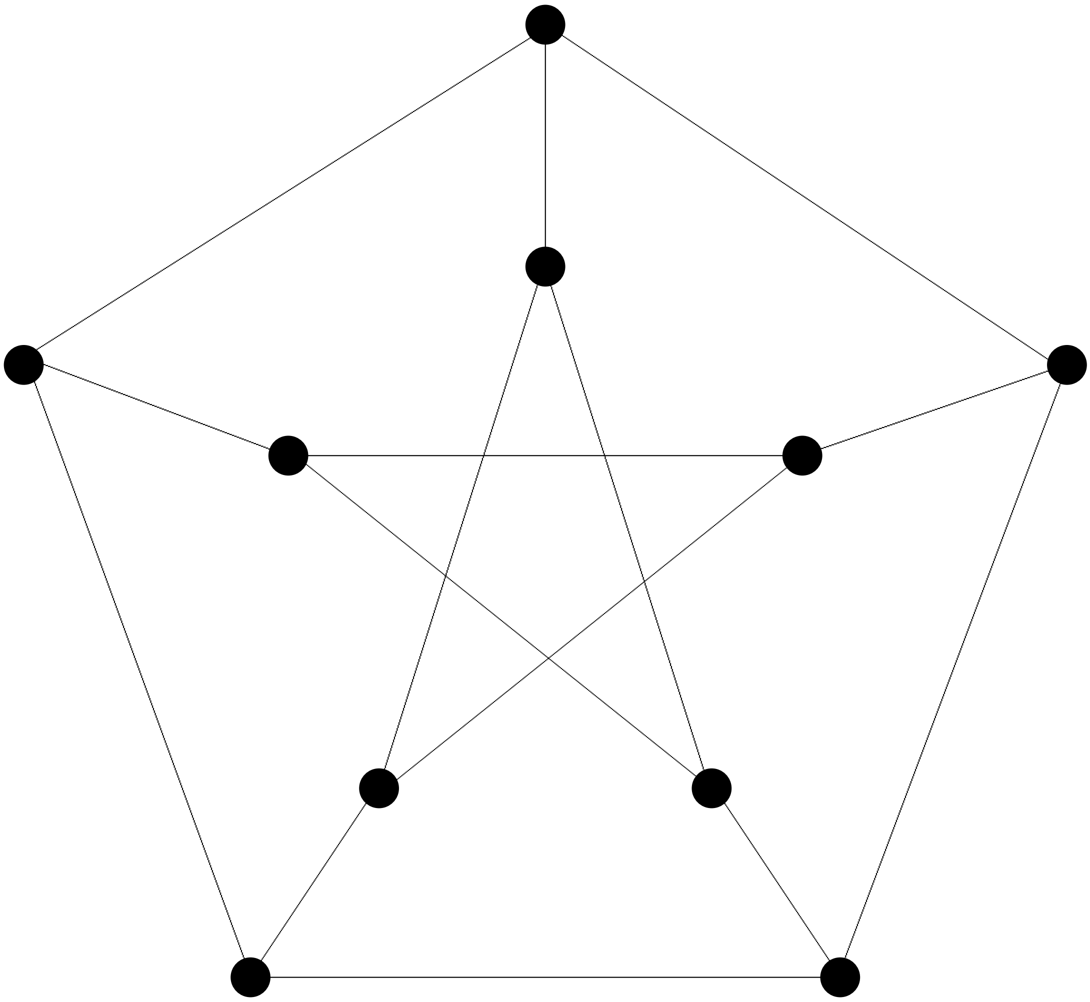

Graph Theory
(Math415)

Problems Booklet

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Math415

Graph Theory

Summer 2003

Time : at least 12 hours

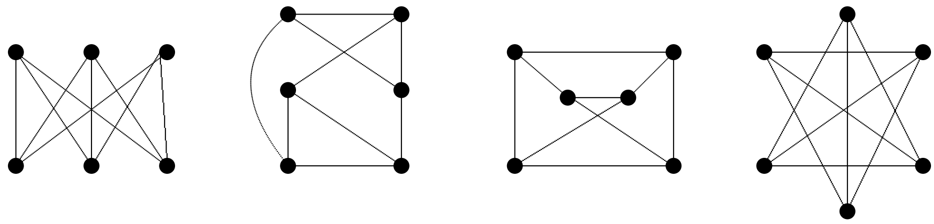
Q1. Draw the following graph:

$$V(G) := \{t, u, v, w, x, y, z\} \text{ and } E(G) := \{uv, tz, vz, xz, yw, ux\}.$$

Which of these vertices are terminal or isolated ?

Q2. Suppose that G is a 3-regular graph with 10 vertices. How many edges does it have ? Does such a G exist with 11 vertices ? Prove that if G is r-regular and r is odd then G has an even number of vertices.

Q3. Which of these graphs are isomorphic and which are not ? (give all your reasoning)



Q4. Test the following sequences using the theorem from the notes to see if they are graphical or not: (if they are graphical draw an exemplar graph having that valency sequence)

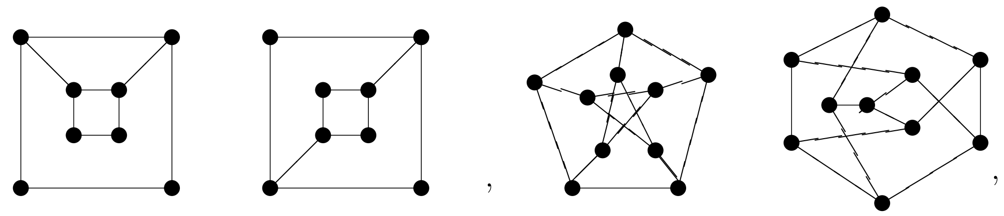
(a) $(5, 5, 3, 3, 2, 2, 2)$

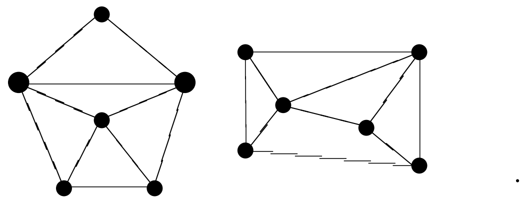
(b) $(8, 6, 5, 4, 3, 2, 2, 2)$

(c) $(4, 3, 3, 3, 2, 2, 2, 1)$

(d) $(5, 4, 3, 2, 1, 1, 1, 1, 1, 1, 1, 1)$

Q5. Prove whether or not the following pairs of graphs in the figure above are isomorphic:





Q6. Which are the eleven graphs on four vertices ? Which of these are *not* bipartite ?

Q7. Find all graphs with valency sequence (4, 2, 2, 2, 1, 1).

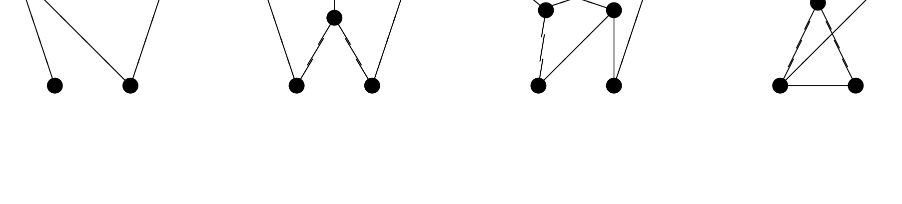
Q8. What is the valency sequence of the graph in question 1 ? How many components does it have ? Are there any different *simple* graphs with the same valency sequence ? Draw those that you find.

Q9. How many vertices and edges are there in $K_{i,j}$ and $K_{i,j,k}$?

Q10. Let $G := K_1 \cup K_2$ and $H := C_4$. Find $G + H$, $H \circ G$, $G \circ H$ and $G \times H$.

Q11. Let G and H have valency sequences $(3, 3, 2, 1, 1)$ and $(4, 2, 2, 2, 1, 1)$. Can you say what the valency sequences of $G + H$, $H \circ G$, $G \circ G$ and $G \times H$ are ?

Q12. Which of the graphs in this figure are bipartite (prove your answers) ?



Q13. If G_i has n_i vertices and m_i edges for $i = 1, \ldots, 3$, how many edges and vertices do these graphs have ?

$$(G_1 \cup G_2) \times G_3 \quad G_1 \circ (G_2 \times G_3) \quad (G_1 + G_2) + G_3 \quad (G_1 \times G_2) + G_3$$

Q14. What are the complements of these graphs ?

$$P_5, \quad C_6, \quad K_3 \times K_2, \quad K_{4,2}$$

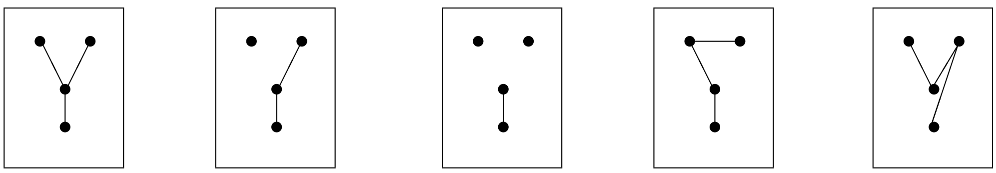
Q15. Using the lemmas in the notes identify the self-complementary graphs on less than four vertices.

Q16. Find a self-complementary graph you haven't already seen.

Q17. Show that a self-complementary graph cannot have a vertex connected to either all or none of the other vertices.

Q18. Prove that regular graphs are reconstructable. [hint: show that the deck of a regular graph is recognizable, and then explain how to reconstruct it]

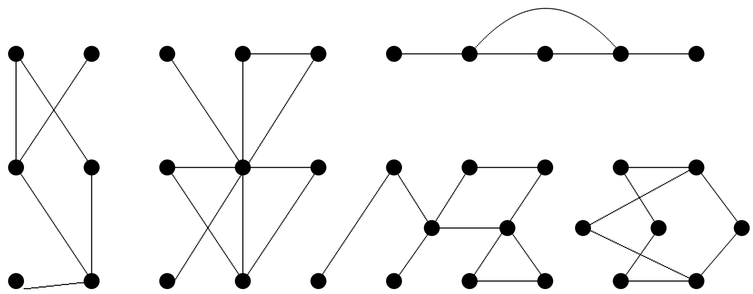
Q19. Which graph has the deck in the figure below ?



Q20. Are there graphs with the following valency sequences ? :
(2n+1, 2n+1, 2n-1, 2n-1, ..., 3, 3, 1, 1),
(2n, 2n-2,..., 4, 3, 2, 2, 1)
(2n, 2n-1, ..., n+1, n, n, n-1, ..., 3, 3, 1, 1) ?

Q21. Prove that if there exists a walk between two vertices then there is a path between them too.

Q22. Show that if G has n vertices and $\min\{\rho(v); v \in V(G)\} \geq (n-1)/2$ then G is connected.



Q23. Find the eccentricities of all the vertices and hence mark the centres and peripheries of the graphs in the figure above.

Q24. If the *square* of a graph G, G^2, is defined as the supergraph of G which has an edge between u and v if and only if u and v are a distance at most two apart in G, find C_5^2, P_6^2, K_n^2 and $K_{m,n}^2$.

Q25. Show that in a connected graph the radius and diameter of a graph G satisfy $r(G) \leq d(G) \leq 2r(G)$. Give a graph for which $r(G) = d(G)$ and another which satisfies $d(G) = 2r(G)$.

Q26. Is it possible that a graph can have vertices of eccentricity $e-1$ and $e+1$ but no vertex of eccentricity e ? [either prove or give a counter-example]

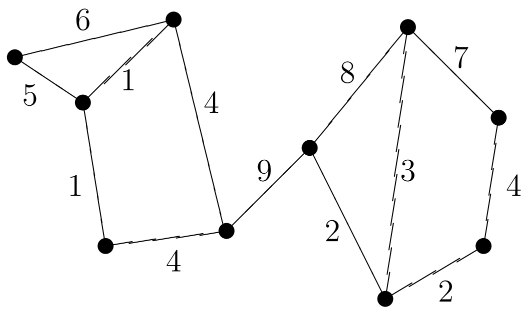
Q27. Find a graph which doesn't have a centre isomorphic to either K_1 or K_2. Is there any limit upon the size of the centre of a graph ? Does the centre of a graph have to be connected ?

Q28. Use Kruskal's algorithm to find a least weight path on the graph in this figure.

Q29. What is the connectivity of $K_{m,n}$, C_n, W_n, Petersen, a path, a tree ?

Q30. [G.A. Dirac, 1952] A finite, simple graph G which has $\rho(v) \geq d \geq 2\ \forall v \in G$ has a circuit of length at least $d+1$.

Q31. If T is a *finite* tree then $|V(G)| = |E(G)| + 1$. (Hint: use induction and a theorem from the notes)



Q32. Identify which graph representing matrix is used in each of these questions and thus draw the graphs represented by these matrices:

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Q33. We define the *line graph*, $L(G)$, of a graph G as the graph formed from G by replacing each edge of G by a vertex and joining two vertices if the two edges they represent were adjacent in G. Find $L(C_5)$, $L(P_5)$ and $L(K_n)$.

Q34. Prove by a constructive argument that K_5 and $K_{3,3}$ are non-planar.

Q35. Show that these graphs are non-planar by finding a subgraph homeomorphic to $K_{3,3}$ in them: Petersen, $P_4 + P_3$, $K_{2,3,2}$.

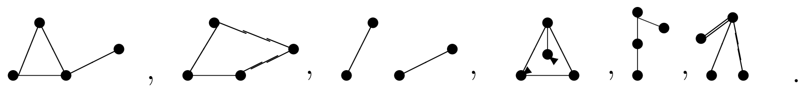
Q36. Embed K_6 and $P_3 \times P_3$ on the projective plane. Clearly mark the faces of each graph and hence count their number.

Q37. Embed K_7, $K_{4,4}$ and $P_4 \times P_3$ on the torus. Find the relationship between numbers of faces, edges and vertices.

Q38. Use the algorithm in the notes to construct a plane embedding of this graph:

$$\begin{aligned} V(G) &:= \{a, b, c, d, e, f, g\} \\ E(G) &:= \{ab, ad, ae, af, bc, bd, be, bg, cd, ce, cf, cg, df, ef, eg\} \end{aligned}$$

Then use the theorem of Wagner to redraw it so that every edge is a straight line but it is still plane.



Q39. Given that the length of the smallest circuit in a planar graph G (its *girth*) is g, prove that $m \leq \frac{g}{g-2}(n-2)$. [hint: use the same ideas as the result $m \leq 3n-6 = 3(n-2)$]

Q40. Draw the five regular polyhedra. Create a new graph from these (their *dual* graph) by replacing each face in each of them by a vertex and joining those vertices whose faces were adjacent in the original graph. To which graphs are the duals isomorphic ?

Q41. What is the (vertex) chromatic number of K_n, $K_{m,n}$, C_n, W_n, a tree and the Petersen graph ? (make sure that the number of colours used is the minimum necessary)
What is the edge chromatic number of these graphs ? (their *chromatic index*)

Q42. Calculate the chromatic polynomials of the graphs in the figure at the top of the page.

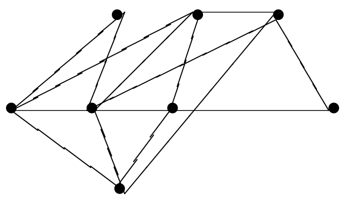
Q43. What are $\chi(C_n)$, $\chi(W_n)$ and $\chi(T)$? (T a tree)
[hint: start from small graphs and try to find a general formula]

Q44. Show that for an n vertex graph G these equations hold:

$$\begin{aligned} 2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) &\leq n+1 \\ \chi(G)\chi(\overline{G}) &\geq n \end{aligned}$$

Q45. If G has n vertices and the size of the largest complete subgraph of $\overline{G}$ is q ($\overline{G}$'s *clique* size) then show that $\chi(G) \geq \frac{n}{q}$.

Q46. If G has n vertices and the smallest valency in G is δ show that $\chi(G) \geq \frac{n}{n-\delta}$.



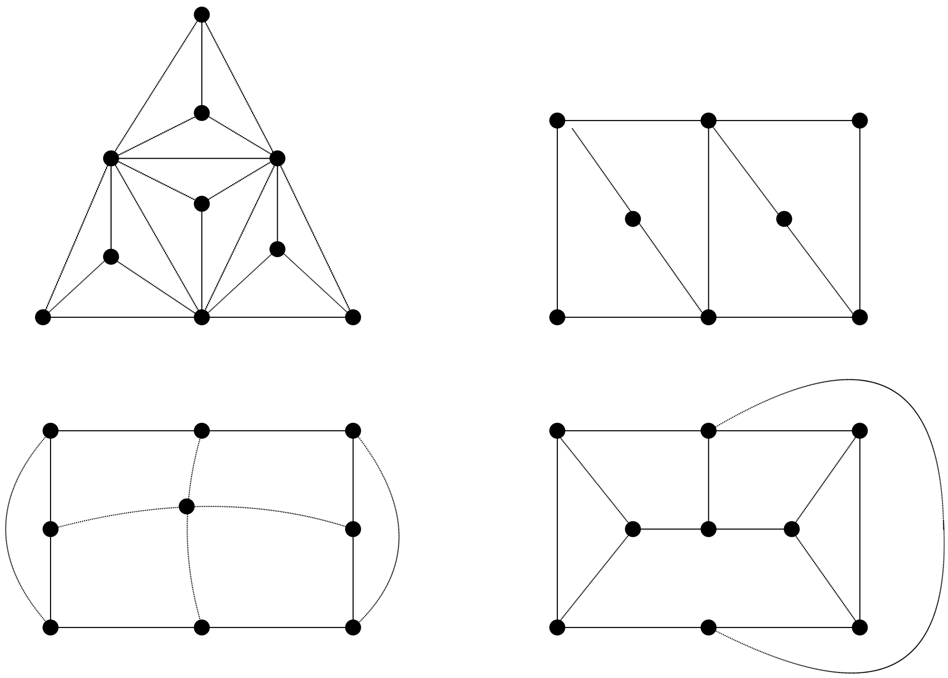
Q47. Prove that Fleury's algorithm does indeed give an Euler-circuit. Use the algorithm to find such a circuit in the graph in the figure above.

Q48. In what circumstances can an arbitrary collection of dominoes be arranged in a line as shown ([1|4] [4|0] [0|0] . . .) ? For which n can the whole set of dominoes labeled with numbers 0 to n be so arranged ? [dominoes are wooden rectangles with two numbers on each end]

Q49. Describe (or draw) a graph on six vertices which is Eulerian and Hamiltonian and describe an Euler circuit and a Hamiltonian circuit in it. Do the same for Eulerian but not Hamiltonian, not Eulerian but Hamiltonian and neither Eulerian nor Hamiltonian.

Q50. Find a graph which isn't Hamiltonian but which has a solution to the Grinberg's theorem equation.

Q51. Which of these graphs are Hamiltonian ? Mark a Hamiltonian path (if one exists) if the graph is not hamiltonian.



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