# Math 4204 Assignment 4: Splitting/NonComm/Modules

Answer all questions and show all working and check each of your results. Any rough work done before attempting your solutions should be attached to your answers as I need to know how you came up with them. You are allowed to talk with myself or other members of the class in genera+l about the questions, but you must do them on your own.

1. Choose an irreducible monic cubic polynomial $p(x) \in \mathbb{Z}_3[x]$ (other than $x^3 + 2x + 2$) which is unique in the class and let $j \notin \mathbb{Z}_3$ be a root of $p(x) = 0$.

   (a) Verify that $\mathbb{Z}_3[j]$ is a field by finding the inverses of each non-zero element. Use pure algebra to find the inverse of at least one quadratic and the Euclidean algorithm on one other and sensible shortcuts (with explanations) where appropriate. [4]

   (b) Find an isomorphism between your field and the field $K := \mathbb{Z}_3[k]$ where $k^3 = k + 1$, verifying that it is preserved under both sum and product. [3]

   (c) Find the splitting field for $p(x)$ over $\mathbb{Z}_3$ and determine its dimension. [3]

2. The numbers represented by $l$ and $m$ should be replaced by the two largest digits of your registration number. Let $X$ be the matrix $\begin{pmatrix} 0 & 0 \\ l & m \end{pmatrix}$ in the non-commutative ring of $2 \times 2$ matrices under $\mathbb{Z}_{11}$.

   (a) Verify that $X$ is both a left zero divisor and a right zero divisor. Determine the set $S$ of matrices that are left zero divisors of $X$. What is the set $T$ of matrices $W$ which satisfy $WX = XW = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$? Are either $S$ or $T$ left or right ideals? [4]

   (b) Explain using algebra why, in a ring with a particular element $a$, the set $D_a := \{y; ay = 0\}$ is necessarily an ideal. Prove that, in a ring, if $a$ is an element with more than one left inverse then $a$ must be a right zero-divisor. [3]

3. Choose a integer $c > 16$ which is not square or square-free (different from all others in the class) and let $A := \mathbb{Z}_c$. Verify what $A(p)$ is for all primes $p$. If $V$ is the module containing all $2 \times 1$ vectors with entries from $A$, describe the elements of the torsion subset of $V$. Is the torsion subset closed under addition? [3]

4. (a) Prove that a finite ring $A$ of characteristic $p$ (an odd prime) is a field if, given any unit $u \in A$, then either $u + 1$ is a unit or is zero. [3]
   [Hint: for $a \in A$ investigate the element $b = a^{p^n - 1}$ and use $b^2 - b$ to find a unit]

   (b) In which place does the proof go wrong if you were to try to use $p = 2$? Give an example of an infinite ring nobody else chooses which satisfies our condition but which is not a field. [2]