University College of Cape Breton

Math421

Modern Algebra I

December 2003

Time : 300 hours

Answer all questions, giving all working and reasoning.

Q1. (a) Given $T := \langle s, t : s^6 = e, sts = t, s^3 = t^2 \rangle$, do coset enumeration with $H = \langle s^2 \rangle$ to find the number of elements in T. [4]

(b) Find the simplest form of each element in terms of s and t and draw a Cayley diagram using the T-relations. [4]

(c) Find the orders of each elements and hence choose elements to be x and y which will satisfy $x^4 = e$, $y^3 = e$ and $yxy = x$ and use Tietze transformations to show the two different presentations are equivalent. [5]

(d) Draw the Cayley diagram after doing coset enumeration on $\langle x \rangle$. Identify the cosets for $\langle x \rangle$ on both diagrams. [4]

(e) Verify that we can take $s := (123456)(789abc)$ and $t := (194c)(285b)(376a)$ which satisfy all three relations, and find permutations which satisfy the relation for x and y. [4]

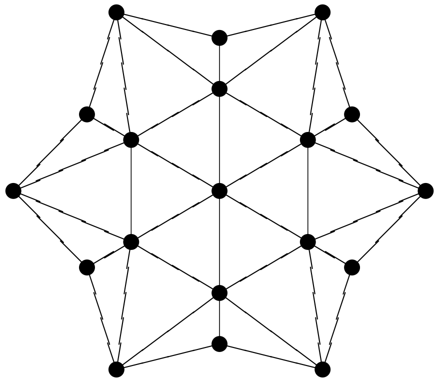
(f) Find the order of $(1743)(25968)$ and count how many conjugates it has in S_{11}. [4]

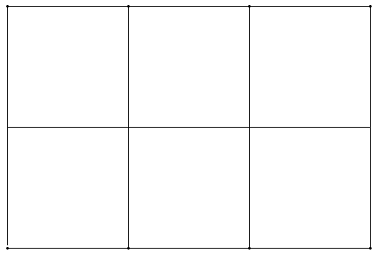
Q2. (a) Given the rectangular pattern below, we want to mark the edges with a thick line. Assuming the pen used shows through to the back of the paper, we want to count the total number of different edge patterns with the possibility that the paper is flipped or rotated.

(i) How many different ways to mark one edge are there? Draw them all. [3]

(ii) How many ways to mark n edges in general? [5]

(iii) If we instead mark the square faces, three green, two red and one blue, calculate the number of such colourings and then draw them all. [5]





(b) Considering the hexagonal figure now, it is on a glass board which can be both flipped and rotated.

(i) If we are marking two vertices and three edges, how many different patterns are there? [4]

(ii) If we have n colours and are colouring the faces, what polynomial represents the number of different colourings? [3]

(iii) If the central hexagon can also be rotated and flipped independently, what is the group formed by this together with the original symmetries? [3]

(iv) Using this group, calculate the number of different ways to colour the faces with n colours. [4]

Q3. Let G be the group $\mathbb{Z}_6 \times \mathbb{Z}_6$, and define the subgroups $H := \langle (0,4), (2,2) \rangle$ $J := \langle (3,1) \rangle$

(a) How many cyclic subgroups of each order are there in G? Which non-cyclic subgroups are have isomorphs in G? [4]

(b) Show that $\phi : (i,j) \to (i,0)$ is a homomorphism and find its kernel. [3]

(c) Verify the first isomorphism theorem such that $\phi(G) \cong G/\ker\phi$. [4]

(d) Calculate $M := H \cap J$ and verify the third isomorphism theorem so that [6]

$$(G/M)/(H/M) \cong G/H.$$

(e) Evaluate HJ and $(HJ)/J$ and show it is isomorphic to H/M, thus verifying the second isomorphism theorem. [5]

(f) Prove that for any two subgroups of a general group, K (which is normal) and H (which is just an ordinary subgroup), that $K \cap H \lhd H$. Give an example of two subgroups in a group whose intersection is not a normal subgroup. [5]

Q4. Let G be the set of all $n \times n$ matrices with non-zero determinant.

(a) Verify that G is a group by verifying the axioms, referencing linear algebra results where necessary, and prove G is not Abelian. [5]

(b) Let U be the subset of G containing just those of determinant +1 or -1. Perform the subgroup and normal subgroup tests on U to show $U \lhd G$. Describe G/U's elements and group operation. [9]

(c) Let V be the set of orthogonal matrices such that $A^{-1} = A^T$ and prove that V is not a normal subgroup of U, but is a proper subgroup of U. [4]

(d) Prove that for any groups such that $R \subseteq S \subseteq T$ such that $R \lhd T$ then $R \lhd S$. [2]

(e) If W is the set of matrices of determinant 1, what is its relation to U and V? [4]

(f) Prove that if $g.x := Ax$ (where $g := A$ is a matrix from G and x is an element of $X := \mathbb{R}^n$) is a group action on X. What is the stabiliser and the orbit of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$? [5]

END OF QUESTION PAPER