HMTH215
MTH215

UNIVERSITY OF ZIMBABWE

Bsc Honours/General Part II/III

Graph Theory

June 2000
Time : 2 hours

Candidates may attempt ALL questions in Section A and at most TWO questions in Section B. Each question should start on a fresh page.

SECTION A (40 marks)

Candidates may attempt ALL questions being careful to number them A1 to A4.

A1. Suppose G is Eulerian. Under what circumstances is $\overline{G}$ (the complement of G) also Eulerian ? Give two n-vertex Hamiltonian graphs G_1 and G_2 such that $\overline{G_1}$ is not Hamiltonian and $\overline{G_2}$ is, and explain why we must specify $n \geq 5$. [10]

A2. What are all the possible valency sequences of trees with 6 vertices ? Give examples of a tree with each sequence. Which of these sequences give a unique tree and which can also arise as the valency sequence of a graph which is not a tree ? [10]

A3. What is the chromatic polynomial of the graph with this adjacency matrix ? [10]

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

A4. Give 2-connected graphs with 9 vertices with matching numbers 2, 3 and 4, proving the connectivity and matching numbers. Why can't there be matching numbers 1 or 5 in such a graph ? [10]

SECTION B (60 marks)

Candidates may attempt TWO questions being careful to number them B5 to B8.

B5. (a) Given a planar graph G, prove that it is possible to add edges to it in order to make every face in the resultant graph T a triangle. Explain why T will have chromatic number at least $\chi(G)$, the chromatic number of G. [8]

(b) Using Euler's formula for connected planar graphs ($n - m + f = 2$) and counting the number of vertices in T of valency i as n_i, prove [8]

$$\sum_{i=1}^{\infty}(6-i)n_i = 12$$

(c) Deduce that there is always a vertex of valency at most 5 in any planar graph. [4]

(d) Using induction on the number of vertices in a graph, prove that it is possible to colour any planar graph with at most five colours, by deleting from the graph the vertex of lowest valency in the inductive case. [10]

B6. (a) Give examples of three different families of graph whose decks are made up of isomorphic cards. [5]

(b) Prove that for any G in the families above that $K_2 \times G$ also has this property. Is this statement also true for $G \oplus G$ and $G \circ K_2$? [15]

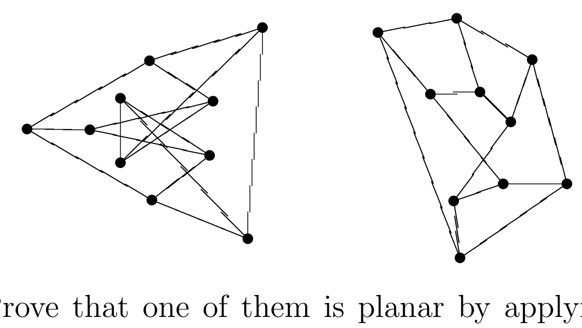
(c) Prove that any such graph is regular and describe how to reconstruct it. [6]

(d) Give an example of a graph which is not in any of the above families which also has the property in question. [4]

B7. (a) Embed $P_4 \times \overline{(P_3 \cup K_1)}$ on the torus, marking the faces and listing the sizes of all the faces. Prove that it cannot be embedded on the plane. [18]

(b) Use the Euler-Poincaré formula to identify which graphs G it is impossible to embed $P_4 \oplus G$ on the surface with Euler-Poincaré characteristic χ. [12]

B8. (a) Show that these two graphs have identical valencies and eccentricities. [7]



(b) Prove that one of them is planar by applying the planar embedding algorithm (in detail, showing all steps used) and the other is non-planar by Kuratowski's theorem and hence deduce that they are not isomorphic. [17]

(c) Are either of these graphs Hamiltonian or bipartite ? [4]

END OF QUESTION PAPER