UNIVERSITY OF ZIMBABWE

HMTH215
MTH215

Bsc Honours Part II, General Part II/III Mathematics

GRAPH THEORY

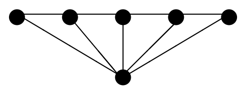
February 1998
Time : 2 hours

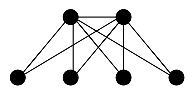
Candidates may attempt ALL questions in Section A and at most TWO questions in Section B. Each question should start on a fresh page.

SECTION A (40 marks)

Candidates may attempt ALL questions being careful to number them A1 to A4.

A1. What are the valency sequences of the two graphs below ? Prove whether each sequence gives rise to a unique graph or not and, if not, display a non-isomorphic (simple) graph which has the same sequence. [10]





A2. Prove that a graph is bipartite if and only if it contains no closed walk of odd length. [10]

A3. What does it mean for a graph to be described as Eulerian ? Hamiltonian ? Draw or describe four graphs on seven or more vertices which are, respectively, both Eulerian *and* Hamiltonian, neither Eulerian *nor* Hamiltonian, Eulerian but *not* Hamiltonian and, finally, *not* Eulerian *but* Hamiltonian. [10]

A4. How many edges and vertices does the complete bipartite graph $K_{m,n}$ have ? Embed $K_{3,4}$ in the torus and clearly indicate and count the faces in the embedding. Which face is bounded by all four vertices of the larger partite set ? Hence or otherwise embed $K_{4,4}$ and verify that the Euler-Poincaré characteristic of the torus is the same. [10]

SECTION B (60 marks)

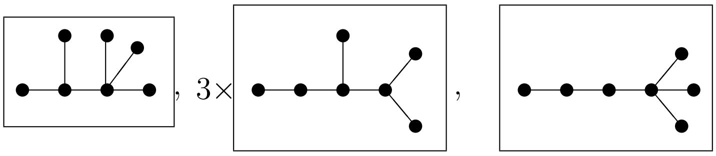
Candidates may attempt TWO questions being careful to number them B5 to B8.

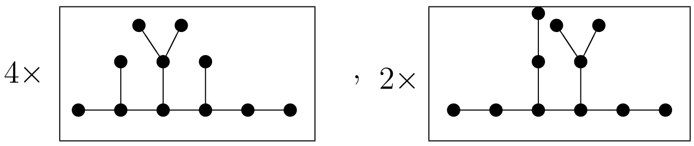
B5. Draw: K_4, $K_{3,3}$, $K_{1,5}+K_3$, $\overline{K_{2,2}\times K_{1,3}}$, $K_2\times K_{1,3}$ $K_{2,3}\circ K_3$ and $(K_2\times K_3)+\overline{K_2}$. [12]

Using Kuratowski's theorem or otherwise identify which of the above graphs are and are not planar, proving the case either way. [18]

B6. (a) Prove that a graph which is connected and contains no circuits has $n-1$ edges and n vertices. Show that, if we are given a deck of a graph G and told G was connected, that we can recognise whether G was or was not a tree. [10]

(b) We define the *end-deck* of G as the set of cards in the deck which correspond to the removal of vertices of valency 1 from G. Given the following two end-decks follow the instructions and hence reconstruct both trees:





(i) Find the diameter and centre of each card. [8]

(ii) Considering only those cards with maximal diameter, look at the structure of each "branch" off the centre and hence reconstruct that tree. [10]

(iii) Construct the end-decks of your answers to verify that you have the correct solution. [2]

B7. What is a self-complementary graph ? Prove that the number of vertices in a self-complementary graph is either $4n$ or $4n+1$, $n\in\mathbb{Z}$. [10]

Exhibit three different self-complementary graphs with eight vertices. [20]

B8. (a) Define the graph theoretical concepts of radius, connectivity and girth. Explain why radius is undefined for disconnected graphs and why $g\geq 3$. [6]

(b) Let G have girth g, n vertices and m edges. Prove that $m\leq \frac{g}{g-2}(n-2)$. [12]

(c) Give examples of graphs with these combinations of the three parameters: [12]

radius	2	6	3	5
connectivity	1	2	3	2
girth	∞	3	3	5

END OF QUESTION PAPER