HMTH215
MTH215

UNIVERSITY OF ZIMBABWE

Bsc Honours/General Part II Supplementary

Graph Theory

July 1999
Time : 2 hours

Candidates may attempt ALL questions in Section A and at most TWO questions in Section B. Each question should start on a fresh page.

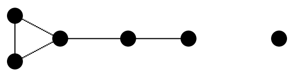
SECTION A (40 marks)

Candidates may attempt ALL questions being careful to number them A1 to A4.

A1. Find two graphs with valency sequence (4,4,3,3,3,3,2,2), one which is planar and one which isn't. Prove the planarity and non-planarity. Find one tree which is a spanning tree for both graphs. [10]

A2. Prove that any Hamiltonian graph has connectivity at least 2. Give a Hamiltonian graph with connectivity 3, proving that no set of two vertices can disconnect it. Give an Eulerian graph with connectivity 1. [10]

A3. We define the incidence matrix as the $m \times n$ matrix M which has an entry 1 in the (i, j) position if edge e_i is incident with vertex v_j. Prove that the adjacency matrix $A = MM^T - D$ where D is a diagonal matrix with the values $\rho(v_i)$ in the (i, i) position.

Verify this relation for this graph:  [10]

A4. What is the chromatic polynomial for $K_{1,n}$ and $K_{2,n}$? Hence, or otherwise, list the different ways to 3-colour $K_{2,3}$, not counting simple permutations of the colours. [10]

SECTION B (60 marks)

Candidates may attempt TWO questions being careful to number them B5 to B8.

B5. (a) Prove Euler's formula for (perhaps disconnected) graphs in the plane. [10]

(b) Deduce the number of faces in a planar drawing of this graph and then list them, indicating how many of each size: [5]

$$\{ab, ad, ae, af, bd, bf, bh, ce, cf, ch, ci, de, dh, di, eg, ei, fh, gi, hi\}$$

(c) Embed this graph on both the projective plane and the torus and hence verify the Euler characteristic for these surfaces. [10]

(d) What are the sizes of each of the faces in all three surfaces ? [5]

B6. (a) Prove that in a graph of radius r and diameter d all possible eccentricities between r and d are represented. [8]

(b) Prove that all eccentricites apart from r have to occur at least twice. [7]

(c) Assuming the two results above, give examples of all six possible combinations of eccentricities in a graph of radius 2 and diameter 4. [15]

B7. (a) Using induction on the number of vertices in a graph, prove that if the maximum valency in a graph is Δ then $\chi(G) \leq \Delta + 1$. [12]

(b) Give a 2-regular graph and a 3-regular graph for which $\chi(G) = \Delta + 1$. [5]

(c) Explain why, in trying to prove that $\chi(G) = \Delta$ for all graphs apart from those in the previous part of the question, it is sufficient to prove it only for G Δ-regular. [5]

(d) Give examples of non-regular graphs with $\chi(G) = \Delta$ for $\Delta = 2, 3, 4$ and 5. [8]

B8. Draw these graphs and answer these questions for each graph in turn, giving all your working and reasons for your answers in each case. [30]

$$P_4 + C_3, \quad C_4 + P_3, \quad W_5 \times (K_1 \cup K_2)$$

(a) Is the graph planar ?

(b) What is its (vertex) connectivity ?

(c) What is its diameter ?

(d) What is its chromatic number ?

(e) What is its periphery ?

(f) Is it Hamiltonian ?

END OF QUESTION PAPER